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www.elsevier.com/locate/jalgebraCocycle deformations for Hopf algebras with a coalgebra projection[☆]Alessandro Ardizzoni^a, Margaret Beattie^{b,*}, Claudia Menini^a^a Department of Mathematics, University of Ferrara, Via Machiavelli 35, Ferrara I-44121, Italy^b Department of Mathematics and Computer Science, Mount Allison University, Sackville, NB E4L 1E6, Canada

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ABSTRACT

Let H be a Hopf algebra over a field K of characteristic 0 and let A be a bialgebra or Hopf algebra such that H is isomorphic to a sub-Hopf algebra of A and there is an H -bilinear coalgebra projection π from A to H which splits the inclusion. Then $A \cong R \#_{\xi} H$ where R is the pre-bialgebra of coinvariants. In this paper we study the deformations of A by an H -bilinear cocycle. If γ is a cocycle for A , then γ can be restricted to a cocycle γ_R for R , and $A^{\gamma} \cong R^{\gamma_R} \#_{\xi_{\gamma}} H$. As examples, we consider liftings of $B(V) \# K[\Gamma]$ where Γ is a finite abelian group, V is a quantum plane and $B(V)$ is its Nichols algebra, and explicitly construct the cocycle which twists the Radford biproduct into the lifting.

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* Corresponding author.

E-mail addresses: alessandro.ardizzoni@unife.it (A. Ardizzoni), mbeattie@mta.ca (M. Beattie), men@dns.unife.it (C. Menini).

URLs: <http://www.unife.it/utenti/alessandro.ardizzoni> (A. Ardizzoni), <http://www.mta.ca/~mbeattie/> (M. Beattie),

<http://www.unife.it/utenti/claudia.menini> (C. Menini).

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1. Introduction

Let A, H be Hopf algebras over a field K of characteristic 0 and suppose that $\sigma : H \hookrightarrow A$ embeds H as a sub-Hopf algebra of A . If there is a Hopf algebra projection $\pi : A \rightarrow H$ such that $\pi \circ \sigma$ is the identity, then A is isomorphic to a Radford biproduct $R \# H$ [Rad] of the algebra of co-invariants $R = A^{\text{co}\pi}$ and the Hopf algebra H . In this setting R is not a sub-Hopf algebra of A but is a Hopf algebra in the Yetter–Drinfeld category ${}^H_H\mathcal{YD}$.

Suppose H is a Hopf algebra over K , A a bialgebra, $\sigma : H \hookrightarrow A$ a bialgebra embedding and π an H -bilinear coalgebra homomorphism from A to H that splits σ . Then the 4-tuple (A, H, π, σ) is called a splitting datum. If π is an algebra homomorphism, then A is a Radford biproduct as above. More generally, $A = R \#_{\xi} H$, where R is the set of π -coinvariants and is a coalgebra in ${}^H_H\mathcal{YD}$ which is not a bialgebra but what was termed in [AMSte] a pre-bialgebra with cocycle ξ . If π is only left H -linear, then π was called a weak projection by Schauenburg [Scha]; he showed that bicrossproducts, double crossproducts and all quantized universal enveloping algebras are examples of this situation.

Using the machinery from [AMSte, AMStu] and [AM], we explore the relationship between the associated pre-bialgebras for a splitting datum (A, H, π, σ) and the splitting datum $(A^{\gamma}, H, \pi, \sigma)$ where A^{γ} is a cocycle deformation of A . Cocycle deformations of Hopf algebras are of interest in the problem of the classification of Hopf algebras. In particular, it has recently been proved (see [GM, Theorem 4.3] and [Mas1, Appendix]) that the families of finite-dimensional pointed Hopf algebras with the same associated graded Hopf algebra $\mathcal{B}(V) \# K[\Gamma]$ classified by Andruskiewitsch and Schneider in [AS1] are cocycle deformations of a Radford biproduct. We define the notion of a cocycle twist for a pre-bialgebra (R, ξ) and show that given a splitting datum as above, if A^{γ} is a cocycle twist of A then $A^{\gamma} \cong R^{\gamma_R} \#_{\xi_{\gamma}} H$ where R^{γ_R} is a cocycle twist of R .

This paper is organized in the following way. In a preliminary section we first recall basic facts about coalgebras in the Yetter–Drinfeld category ${}^H_H\mathcal{YD}$, prove some key lemmas, and review the basic theory of pre-bialgebras with a cocycle in ${}^H_H\mathcal{YD}$ from [AMSte] and [AMStu], ending with some examples. In general a pre-bialgebra with cocycle (R, ξ) does not have associative multiplication for ξ nontrivial. In Section 3 we show that if R is connected, the sufficient conditions for (R, ξ) to have associative multiplication from Section 2.2.1 are also necessary. Section 4 contains the main results of this paper. Here we review the notion of a cocycle twist for a Hopf algebra A , and define cocycle twists for pre-bialgebras. We show that for (R, ξ) a pre-bialgebra with cocycle associated to a splitting datum (A, H, π, σ) , then the set of H -bilinear cocycles on A is in one–one correspondence with the left H -linear cocycles on R . Furthermore, a cocycle twist of A , say $A^{\gamma} \cong (R \#_{\xi} H)^{\gamma}$, is isomorphic to $R^{\gamma_R} \#_{\xi_{\gamma}} H$ where γ_R is the cocycle on $R \otimes R$ corresponding to the cocycle γ for A and $(R^{\gamma_R}, \xi_{\gamma})$ is the pre-bialgebra with cocycle corresponding to A^{γ} . In Section 5, we explicitly describe the cocycle which twists the Radford product $\mathcal{B}(V) \# K[\Gamma]$ of the group algebra of a finite abelian group and the Nichols algebra of a quantum plane to the liftings of this pointed Hopf algebra. Examples include the three families of non-isomorphic pointed Hopf algebras of dimension 32 described in [G] and the pointed Hopf algebras of dimension 81 which were among the first counterexamples to Kaplansky's Tenth Conjecture.

Throughout H will denote a Hopf algebra over a field K . All maps are assumed to be over K . We assume for simplicity of the exposition that our ground field K has characteristic zero. Anyway we point out that many results below are valid under weaker hypotheses.

2. Preliminaries

We will use the Heyneman–Sweedler notation for the comultiplication in a K -coalgebra C but with the summation sign omitted, namely $\Delta(x) = x_{(1)} \otimes x_{(2)}$ for $x \in C$. For C a coalgebra and A an algebra the convolution multiplication in $\text{Hom}(C, A)$ will be denoted $*$. Composition of functions will be denoted by \circ or possibly by juxtaposition when the meaning is clear.

A Hopf algebra H is a left H -module under the adjoint action $h \rightarrow m = h_{(1)}mS(h_{(2)})$ and has a similar right adjoint action. Recall [AMSte, Definition 2.7] that a left and right integral $\lambda \in H^*$ for H is called ad-invariant if $\lambda(1) = 1$ and λ is a left and right H -module map with respect to the left and right adjoint actions. If H is semisimple and cosemisimple, then the total integral for H is ad-invariant; see either [SvO, Proposition 1.12, b)] or [AMSte, Theorem 2.27]. If H has an ad-invariant integral, then H is cosemisimple.

We assume familiarity with the general theory of Hopf algebras; good references are [Sw, Mo].

2.1. Coalgebras in a category of Yetter–Drinfeld modules

Let H be a Hopf algebra over K . Coalgebras in ${}^H_H\mathcal{YD}$, the category of left–left Yetter–Drinfeld modules over H , will play a central role in this paper. For (V, \cdot) a left H -module, we write $h \cdot v$ for $h \cdot v$, the action of h on v , if the meaning is clear. The left H -module H with the left adjoint action is denoted (H, \rightharpoonup) ; the left and right actions of H on H induced by multiplication will be denoted by juxtaposition. For (V, ρ) a left H -comodule, for $v \in V$ we write $\rho(v) = v_{(-1)} \otimes v_{(0)}$ for the coaction. Recall that if V is a left H -module and a left H -comodule, then V is an object in ${}^H_H\mathcal{YD}$ if for all $v \in V, h \in H$,

$$\rho(h \cdot v) = h_{(1)}v_{(-1)}S(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}.$$

The field K is a left–left Yetter–Drinfeld module with $\rho(1) = 1 \otimes 1$ and trivial left H action. As well, $(H, \rightharpoonup, \Delta_H)$ is a left–left Yetter–Drinfeld module. If V, W are objects in ${}^H_H\mathcal{YD}$, so is $V \otimes W$ with H action given by $h(r \otimes t) = h_{(1)}r \otimes h_{(2)}t$, and H -coaction given by $\rho(r \otimes t) = r_{(-1)}t_{(-1)} \otimes r_{(0)} \otimes t_{(0)}$ for all $r \in V, t \in W, h \in H$. A map $f \in \text{Hom}(V, W)$ is called (left) H -linear if $f(h \cdot v) = h \cdot f(v)$, for all $h \in H, v \in V$. For example $f \in \text{Hom}(V, K)$ is left H -linear if $f(h \cdot v) = \varepsilon(h)f(v)$ and $f \in \text{Hom}(V, H)$ is left H -linear if $f(h \cdot v) = h_{(1)}f(v)S(h_{(2)})$. The category ${}^H_H\mathcal{YD}$ is braided with braiding $c_{V,W} : V \otimes W \rightarrow W \otimes V$ given by $c_{V,W}(v \otimes w) = v_{(-1)}w \otimes v_{(0)}$.

For C a coalgebra in ${}^H_H\mathcal{YD}$, we use a modified version of the Heyneman–Sweedler notation, writing superscripts instead of subscripts, so that comultiplication is written

$$\Delta_C(x) = \Delta(x) = x^{(1)} \otimes x^{(2)}, \quad \text{for every } x \in C.$$

If C and D are coalgebras in ${}^H_H\mathcal{YD}$, so is $C \otimes D$ defined as follows. As a Yetter–Drinfeld module, $C \otimes D = C \otimes D$ with H -action and coaction as described above. The counit is $\varepsilon_{C \otimes D} = \varepsilon_C \otimes \varepsilon_D$ and the comultiplication is $\Delta_{C \otimes D} = (C \otimes c_{C,D} \otimes D) \circ (\Delta_C \otimes \Delta_D)$, so that

$$\begin{aligned} \Delta_{C \otimes D}(x \otimes y) &= (x^{(1)} \otimes x_{(-1)}^{(2)}y^{(1)}) \otimes (x_{(0)}^{(2)} \otimes y^{(2)}), \\ \Delta_{C \otimes D \otimes E}(x \otimes y \otimes z) &= (x^{(1)} \otimes x_{(-2)}^{(2)}y^{(1)} \otimes x_{(-1)}^{(2)}y_{(-1)}^{(2)}z^{(1)}) \otimes (x_{(0)}^{(2)} \otimes y_{(0)}^{(2)} \otimes z^{(2)}). \end{aligned}$$

When it is clear from the context (and from the superscript versus subscript notation) \otimes is written simply as \otimes .

For a K -coalgebra C and a map $u_C : K \rightarrow C$, the coalgebra (C, u_C) is called *coaugmented* if $1_C := u_C(1_K)$ is a grouplike element. For C a coalgebra in ${}^H_H\mathcal{YD}$, then u_C is also required to be a map in

the Yetter–Drinfeld category, i.e., for all $h \in H$,

$$h \cdot 1_C = \varepsilon_H(h)1_C \quad \text{and} \quad \rho_C(1_C) = 1_H \otimes 1_C. \quad (1)$$

A coaugmented coalgebra C is called connected if $C_0 = K1_C$.

The next definitions and lemmas will be key in later computations.

Definition 2.1. For M a left H -comodule, define $\Psi : \text{Hom}(M, K) \rightarrow \text{Hom}^{H, -}(M, H)$, the left H -comodule maps from M to H , by

$$\Psi(\alpha) = (H \otimes \alpha)\rho_M.$$

Remark 2.2. Let $f : M \rightarrow L$ be a morphism of left H -comodules and $\alpha \in \text{Hom}(L, K)$. Then $\Psi(\alpha) \circ f = \Psi(\alpha \circ f)$. To see this, let $m \in M$ and then

$$(\Psi(\alpha) \circ f)(m) = \Psi(\alpha)(f(m)) = f(m)_{(-1)}\alpha(f(m)_{(0)}) = m_{(-1)}\alpha(f(m_{(0)})) = \Psi(\alpha \circ f)(m).$$

Lemma 2.3. For C a left H -comodule coalgebra, $\Psi : \text{Hom}(C, K) \rightarrow \text{Hom}^{H, -}(C, H)$ is an algebra isomorphism.

Proof. Let ${}^H\mathfrak{M}$ denote the monoidal category of left H -comodules. Then the functor $(-) \otimes H : \text{Vec}(K) \rightarrow {}^H\mathfrak{M}$ is right adjoint to the forgetful functor. The canonical isomorphism defining the adjunction yields Ψ . Explicitly, let $\alpha, \beta \in \text{Hom}(C, K)$ and let $z \in C$. We have

$$\begin{aligned} [\Psi(\alpha) * \Psi(\beta)](z) &= [(H \otimes \alpha)\rho_C] * [(H \otimes \beta)\rho_C](z) \\ &= (H \otimes \alpha)\rho_C(z^{(1)})[(H \otimes \beta)\rho_C](z^{(2)}) \\ &= z_{(-1)}^{(1)}\alpha(z_{(0)}^{(1)})z_{(-1)}^{(2)}\beta(z_{(0)}^{(2)}) \\ &= z_{(-1)}^{(1)}z_{(-1)}^{(2)}\alpha(z_{(0)}^{(1)})\beta(z_{(0)}^{(2)}) \\ &= z_{(-1)}\alpha[(z_{(0)})^{(1)}]\beta[(z_{(0)})^{(2)}] \\ &= z_{(-1)}(\alpha * \beta)(z_{(0)}) = \Psi(\alpha * \beta)(z) \end{aligned}$$

and also $\Psi(\varepsilon_C)(z) = (H \otimes \varepsilon_C)\rho_C = u_H\varepsilon_C$. Thus Ψ is an algebra homomorphism.

For $\alpha \in \text{Hom}(C, K)$, then $\Psi(\alpha)$ is a morphism in ${}^H\mathfrak{M}$ since

$$\Delta_H \Psi(\alpha)(z) = \sum \Delta_H(z_{(-1)})\alpha(z_{(0)}) = \sum z_{(-2)} \otimes z_{(-1)}\alpha(z_{(0)}) = \sum z_{(-1)} \otimes \Psi(z_{(0)}).$$

Finally, the composition inverse of Ψ is Ψ^{-1} where $\Psi^{-1}(\sigma) := \varepsilon_H\sigma$. \square

Remark 2.4. (i) Let $(C, \Delta_C, \varepsilon_C, u_C)$ be a coaugmented connected coalgebra, A an algebra and $f : C \rightarrow A$ a map such that $f(1_C) = 1_A$, i.e., $f = u_A\varepsilon_C$ on the coradical of C . Then by [Mo, Lemma 5.2.10], f is convolution invertible with inverse $\sum_{n=0}^{\infty} \gamma^n$ where $\gamma = u_A\varepsilon_C - f$. Then since $\gamma^{n+1} = 0$ on C_n , it is also true that $f^{-1} = u_A\varepsilon_C$ on C_0 .

(ii) For C a left H -module coalgebra and $v : C \rightarrow K$ convolution invertible, then v is left H -linear if and only if v^{-1} is. For suppose v is left H -linear. Then

$$\nu^{-1}(hz) = \nu^{-1}(hz^{(1)})\nu(z^{(2)})\nu^{-1}(z^{(3)}) = \nu^{-1}(h_{(1)}z^{(1)})\nu(h_{(2)}z^{(2)})\nu^{-1}(z^{(3)}) = \varepsilon_H(h)\nu^{-1}(z),$$

and so ν^{-1} is left H -linear also.

The next condition is part of the definition of a cocycle ξ for a pre-bialgebra [AMStu] (see Section 2.2.1) but makes sense for any coalgebra C in ${}^H_H\mathcal{YD}$.

Definition 2.5. Let C be a coalgebra in ${}^H_H\mathcal{YD}$ and $\alpha \in \text{Hom}(C, H)$. Then we say that α is a dual normalized Sweedler 1-cocycle if $\Delta_H\alpha = (m_H \otimes \alpha)(\alpha \otimes \rho_C)\Delta_C$ and $\varepsilon_H\alpha = \varepsilon_C$. Thus for $x \in C$,

$$\alpha(x)_{(1)} \otimes \alpha(x)_{(2)} = \alpha(x^{(1)})x^{(2)}_{(-1)} \otimes \alpha(x^{(2)}_{(0)}) \quad \text{and} \quad \varepsilon_H(\alpha(x)) = \varepsilon_C(x). \quad (2)$$

If $\alpha : C \rightarrow H$ is a dual normalized Sweedler 1-cocycle, then α is convolution invertible and its inverse can be described explicitly.

Proposition 2.6. Let C be a coalgebra in ${}^H_H\mathcal{YD}$ and let $\alpha : C \rightarrow H$ satisfy (2). Then α' is the convolution inverse of α where

$$\alpha' := m_H \circ (H \otimes S_H \circ \alpha) \circ \rho_C.$$

Proof. For any $x \in C$, we have

$$\begin{aligned} \alpha * \alpha'(x) &= \alpha(x^{(1)})\alpha'(x^{(2)}) = \alpha(x^{(1)})x^{(2)}_{(-1)} S_H(\alpha(x^{(2)}_{(0)})) \stackrel{(2)}{=} \alpha(x)_{(1)} S_H(\alpha(x)_{(2)}) \\ &= \varepsilon_H \circ \alpha(x) \stackrel{(2)}{=} u_H \circ \varepsilon_C(x), \end{aligned}$$

and

$$\begin{aligned} \alpha' * \alpha(x) &= \alpha'(x^{(1)})\alpha(x^{(2)}) = x^{(1)}_{(-1)} S_H(\alpha(x^{(1)}_{(0)}))\alpha(x^{(2)}) \\ &= x^{(1)}_{(-1)} x^{(2)}_{(-2)} S_H(x^{(2)}_{(-1)}) S_H(\alpha((x^{(1)}_{(0)})))\alpha((x^{(2)}_{(0)})) \\ &= x_{(-1)} S_H((x^{(2)}_{(0)})_{(-1)}) S_H(\alpha(x^{(1)}_{(0)}))\alpha((x^{(2)}_{(0)})) \\ &= x_{(-1)} S_H[\alpha(x^{(1)}_{(0)})(x^{(2)}_{(0)})_{(-1)}]\alpha((x^{(2)}_{(0)})) \\ &\stackrel{(2)}{=} x_{(-1)} S_H(\alpha(x_{(0)})(1))\alpha(x_{(0)}(2)) \\ &= x_{(-1)} \varepsilon_H(\alpha(x_{(0)})) = x_{(-1)} \varepsilon_C(x_{(0)}) \\ &= u_H \circ \varepsilon_C(x). \end{aligned}$$

Thus α' is the convolution inverse of α as claimed. \square

The next definition/lemma will be useful in upcoming computations.

Lemma 2.7. Let C be a coalgebra and let (M, μ) be a left H -module. Define

$$\Phi : \text{Hom}(C, H) \rightarrow \text{End}(C \otimes M) \quad \text{by} \quad \Phi(\alpha) := (C \otimes \mu_M)[(C \otimes \alpha)\Delta_C \otimes M],$$

for $\alpha \in \text{Hom}(C, H)$. The map Φ is an algebra homomorphism.

Proof. By definition,

$$\Phi(\alpha)(x \otimes m) = x_{(1)} \otimes \alpha(x_{(2)})m, \quad \text{for } x \in C, m \in M.$$

Then for $\alpha, \beta \in \text{Hom}(C, H)$,

$$\begin{aligned} (\Phi(\alpha) \circ \Phi(\beta))(x \otimes m) &= \Phi(\alpha)(x_{(1)} \otimes \beta(x_{(2)})m) = x_{(1)} \otimes \alpha(x_{(2)})\beta(x_{(3)})m \\ &= x_{(1)} \otimes (\alpha * \beta)(x_{(2)})m = \Phi(\alpha * \beta)(x \otimes m) \end{aligned}$$

and

$$\Phi(u_H \circ \varepsilon_C)(x \otimes m) = x_{(1)} \otimes (u_H \circ \varepsilon_C)(x_{(2)})m = x \otimes m. \quad \square$$

Remark 2.8. Note that $\Phi(\alpha) = \text{Id}_{C \otimes M}$ if and only if the action of $\alpha(C)$ on M is trivial. For if $x \otimes m = x_{(1)} \otimes \alpha(x_{(2)})m$, apply ε_C to the left-hand tensorand to see that $\alpha(C)$ acts trivially on M . The converse is obvious.

2.2. Pre-bialgebras with cocycle

In this section, we recall the notion of a pre-bialgebra with cocycle in ${}^H_H\mathcal{YD}$ and explain how a pre-bialgebra with cocycle is associated to a splitting datum. Throughout this section, H will denote a Hopf algebra over K .

Definition 2.9. (Cf. [AM, Definition 1.8].) A *splitting datum* (A, H, π, σ) consists of a bialgebra A , a bialgebra homomorphism $\sigma : H \rightarrow A$ and an H -bilinear coalgebra homomorphism $\pi : A \rightarrow H$ such that $\pi\sigma = \text{Id}_H$. Note that H -bilinear here means $\pi(\sigma(h)x\sigma(h')) = h\pi(x)h'$ for all $h, h' \in H$ and $x \in A$. We say that a splitting datum is *trivial* whenever π is a bialgebra homomorphism.

Example 2.10. Let H be a Hopf algebra and let $A = R \# H$ be the usual Radford biproduct of a bialgebra R in ${}^H_H\mathcal{YD}$ and H . Let $\sigma : H \rightarrow A$, $\sigma(h) = 1_R \# h$, and let $\pi : A \rightarrow H$, $\pi(r \# h) = r\varepsilon_H(h)$. Then (A, H, π, σ) is a trivial splitting datum. Conversely, for A a Hopf algebra, if (A, H, π, σ) is a trivial splitting datum, then A is isomorphic to a Radford biproduct or bosonization of H (identified with $\sigma(H)$) and the K -algebra R of π -coinvariants, a Hopf algebra in the category ${}^H_H\mathcal{YD}$.

If π is not an algebra map, then $R = \{x \in A \mid (A \otimes \pi)\Delta(x) = x \otimes 1\}$ need not be a Hopf algebra in ${}^H_H\mathcal{YD}$ but instead will be a *pre-bialgebra with a cocycle* in ${}^H_H\mathcal{YD}$.

2.2.1. Definition of a pre-bialgebra and a pre-bialgebra with cocycle

Following [AMStu, Definition 2.3, Definitions 3.1], we define the following. A pre-bialgebra $R = (R, m_R, u_R, \Delta_R, \varepsilon_R)$ in ${}^H_H\mathcal{YD}$ is a coaugmented coalgebra $(R, \Delta_R, \varepsilon_R, u_R)$ in the category ${}^H_H\mathcal{YD}$ together with a left H -linear map $m_R : R \otimes R \rightarrow R$ such that m_R is a coalgebra homomorphism, i.e.,

$$\Delta_R m_R = (m_R \otimes m_R) \Delta_{R \otimes R} \quad \text{and} \quad \varepsilon_R m_R = m_K(\varepsilon_R \otimes \varepsilon_R), \quad (3)$$

and u_R is a unit for m_R , i.e.,

$$m_R(R \otimes u_R) = R = m_R(u_R \otimes R). \quad (4)$$

When clear from the context, the subscript R on the structure maps above is omitted.

Essentially a pre-bialgebra differs from a bialgebra in ${}^H_H\mathcal{YD}$ in that *the multiplication need not be associative and need not be a morphism of H -comodules*, see Example 3.4.

A pre-bialgebra with cocycle in ${}^H_H\mathcal{YD}$ is a pair (R, ξ) where $R = (R, m, u, \Delta, \varepsilon)$ is a pre-bialgebra in ${}^H_H\mathcal{YD}$ and $\xi : C = R \otimes R \rightarrow H$ is a normalized dual Sweedler 1-cocycle (2), left H -linear with respect to the left adjoint action of H on H , and, for all $r, s \in R$ and $h \in H$, the following hold:

$$c_{R,H}(m \otimes \xi) \Delta_{R \otimes R} = (m_H \otimes m_R)(\xi \otimes \rho_{R \otimes R}) \Delta_{R \otimes R}, \quad (5)$$

$$m_R(R \otimes m_R) = m_R(R \otimes \mu_R)[(m_R \otimes \xi) \Delta_{R \otimes R} \otimes R] = m_R(m_R \otimes R) \Phi(\xi), \quad (6)$$

$$m_H(\xi \otimes H)[R \otimes (m_R \otimes \xi) \Delta_{R \otimes R}] = m_H(\xi \otimes H)(R \otimes c_{H,R})[(m_R \otimes \xi) \Delta_{R \otimes R} \otimes R], \quad (7)$$

$$\xi(R \otimes u) = \xi(u \otimes R) = \varepsilon 1_H. \quad (8)$$

By definition [AM], a map $f : ((R, m, u, \delta, \varepsilon), \xi) \rightarrow ((R', m', u', \delta', \varepsilon'), \xi')$ of pre-bialgebras with a cocycle in ${}^H_H\mathcal{YD}$, is a morphism of pre-bialgebras with cocycle if f is a coalgebra homomorphism $f : (R, \delta, \varepsilon) \rightarrow (R', \delta', \varepsilon')$ in the category $({}^H_H\mathcal{YD}, \otimes, K)$ such that

$$f \circ m = m' \circ (f \otimes f), \quad f \circ u = u' \quad \text{and} \quad \xi' \circ (f \otimes f) = \xi.$$

Remark 2.11. In (6) the map Φ from Lemma 2.7 is used with $C = R \otimes R$ and $M = R$. Thus if $\Phi(\xi)$ is the identity, or, equivalently, if $\xi(R \otimes R) \subseteq H$ acts trivially on R , then m_R is associative. We will see in Section 3 that if R is connected, the converse holds.

Let (R, ξ) be a pre-bialgebra with cocycle in ${}^H_H\mathcal{YD}$. Since $\xi : C = R \otimes R \rightarrow H$ is a dual Sweedler 1-cocycle, by Proposition 2.6, ξ is convolution invertible with convolution inverse

$$\xi^{-1} = m_H \circ (H \otimes S_H \circ \xi) \circ \rho_{R \otimes R}. \quad (9)$$

Thus Eq. (6) is equivalent to:

$$m_R \circ (m_R \otimes R) = m_R \circ (R \otimes m_R) \circ \Phi(\xi^{-1}).$$

If, as well, R , and thus $R \otimes R$ is connected, since $\xi(1_{R \otimes R}) = u_H(1_K)$, by Remark 2.4(i) with $C = R \otimes R$ and $A = H$, then another form for ξ^{-1} is

$$\xi^{-1} = \sum_{n=0}^{\infty} (u_H \circ \varepsilon_C - \xi)^n.$$

2.2.2. The splitting datum associated to a pre-bialgebra with cocycle

To every (R, ξ) , we can associate a splitting datum $(A := R \#_{\xi} H, H, \pi, \sigma)$ where the bialgebra $R \#_{\xi} H$ is constructed as follows (see [AMSte, Theorem 3.62] and [AMStu, Definitions 3.1]). As a vector space, $A = R \otimes H$ with coalgebra and algebra structures given below.

Let $r, s \in R, h, h' \in H$. The coalgebra structures are $\varepsilon_A(r \# h) = \varepsilon_R(r) \varepsilon_H(h)$, and

$$\Delta_A(r \# h) = r^{(1)} \# r_{(-1)}^{(2)} h_{(1)} \otimes r_{(0)}^{(2)} \# h_{(2)}, \quad \text{where } \Delta_R(r) = r^{(1)} \otimes r^{(2)}. \quad (10)$$

In other words, as a coalgebra, A is the smash coproduct of R and H .

For future calculations, we note:

$$\Delta_A(r \# 1_H) = r^{(1)} \# r_{(-1)}^{(2)} \otimes r_{(0)}^{(2)} \# 1_H, \quad (11)$$

$$\Delta_A^2(r \# 1_H) = r^{(1)} \# r_{(-1)}^{(2)} r_{(-2)}^{(3)} \otimes r_{(0)}^{(2)} \# r_{(-1)}^{(3)} \otimes r_{(0)}^{(3)} \# 1_H. \quad (12)$$

The unit is $u_A(1) = 1_R \# 1_H$ and the multiplication is given by

$$m_A = (R \otimes m_H)[(m_R \otimes \xi) \Delta_{R \otimes R} \otimes m_H](R \otimes c_{H,R} \otimes H)$$

so that for $r, s \in R, h, h' \in H$,

$$m_A(r \# h \otimes s \# h') = m_R(r^{(1)} \otimes r_{(-1)}^{(2)}(h_{(1)}s)^{(1)}) \# \xi(r_{(0)}^{(2)} \otimes (h_{(1)}s)^{(2)})h_{(2)}h'. \quad (13)$$

Using the map Φ from Lemma 2.7 with $C = R \otimes R$ and $M = (H, m_H)$, we write:

$$\begin{aligned} m_{R \#_{\xi} H} &= (m_R \otimes m_H) \circ (\Phi(\xi) \otimes H) \circ (R \otimes c_{H,R} \otimes H) \\ &= (m_R \otimes H) \circ \Phi(\xi) \circ (R \otimes R \otimes m_H) \circ (R \otimes c_{H,R} \otimes H). \end{aligned} \quad (14)$$

Here, unless $\xi(R \otimes R) = K$, the action of $\xi(R \otimes R)$ will not be trivial and $\Phi(\xi)$ will not be the identity. It will be useful to have the following formulas:

$$(R \otimes \varepsilon_H)m_A(r \# h \otimes s \# h') = m_R(r \otimes hs)\varepsilon(h'), \quad (15)$$

$$(\varepsilon_R \otimes H)m_A(r \# h \otimes s \# h') = \xi(r \otimes h_{(1)}s)h_{(2)}h'. \quad (16)$$

Note that the canonical injection $\sigma : H \hookrightarrow R \#_{\xi} H$ is a bialgebra homomorphism. Furthermore

$$\pi : R \#_{\xi} H \rightarrow H : r \# h \mapsto \varepsilon(r)h$$

is an H -bilinear coalgebra retraction of σ .

2.2.3. The pre-bialgebra with cocycle associated to a splitting datum

Suppose that (A, H, π, σ) is a splitting datum. In this subsection we describe (R, ξ) , the associated pre-bialgebra with cocycle in ${}^H_H\mathcal{YD}$ [AMStu, Definitions 3.2]. As when π is a bialgebra morphism and A is a Radford biproduct, set

$$R = A^{co\pi} = \{a \in A \mid a_{(1)} \otimes \pi(a_{(2)}) = a \otimes 1_H\},$$

and let

$$\tau : A \rightarrow R, \quad \tau(a) = a_{(1)}\sigma S\pi(a_{(2)}).$$

Define a left-left Yetter-Drinfeld structure on R by

$$h \cdot r = hr = \sigma(h_{(1)})r\sigma S_H(h_{(1)}), \quad \rho(r) = \pi(r_{(1)}) \otimes r_{(2)},$$

and define a coalgebra structure in ${}^H_H\mathcal{YD}$ on R by

$$\Delta_R(r) = r^{(1)} \otimes r^{(2)} = r_{(1)}\sigma S\pi(r_{(2)}) \otimes r_{(3)} = \tau(r_{(1)}) \otimes r_{(2)}, \quad \varepsilon = \varepsilon_{A|R}. \quad (17)$$

The map

$$\omega : R \otimes H \rightarrow A, \quad \omega(r \otimes h) = r\sigma(h)$$

is an isomorphism of K -vector spaces, the inverse being defined by

$$\omega^{-1} : A \rightarrow R \otimes H, \quad \omega^{-1}(a) = a_{(1)}\sigma S_H\pi(a_{(2)}) \otimes \pi(a_{(3)}) = \tau(a_{(1)}) \otimes \pi(a_{(2)}).$$

Clearly A defines, via ω , a bialgebra structure on $R \otimes H$ that will depend on σ and π . As shown in [Scha, 6.1] and [AMSte, Theorem 3.64], $(R, m, u, \Delta, \varepsilon)$ is a pre-bialgebra in ${}^H_H\mathcal{YD}$ with cocycle ξ where the maps $u : K \rightarrow R$ and $m : R \otimes R \rightarrow R$, are defined by

$$u = u_A^{|R|}, \quad m(r \otimes s) = r_{(1)}s_{(1)}\sigma S\pi(r_{(2)}s_{(2)}) = \tau(r \cdot_A s)$$

and the cocycle $\xi : R \otimes R \rightarrow H$ is the map defined by

$$\xi(r \otimes s) = \pi(r \cdot_A s).$$

Then (R, ξ) is the pre-bialgebra with cocycle in ${}^H_H\mathcal{YD}$ associated to (A, H, π, σ) .

We note that the map τ above is a surjective coalgebra homomorphism and satisfies the following where $a \in A$, $h \in H$ and $r, s \in R$ [AMStu, Proposition 3.4]:

$$\begin{aligned} \tau[a\sigma(h)] &= \tau(a)\varepsilon_H(h), & \tau[\sigma(h)a] &= h \cdot \tau(a), \\ r \cdot_R s &= \tau(r \cdot_A s), & \tau(a) \cdot_R \tau(b) &= \tau[\tau(a) \cdot_A b]. \end{aligned}$$

Note that $h \cdot r = \tau(\sigma(h)r)$ for all $h \in H$, $r \in R$.

2.2.4. The correspondence between splitting data and pre-bialgebras with cocycle

If we start with a splitting datum (A, H, π, σ) , and construct (R, ξ) as in Section 2.2.3, and then construct the splitting datum $(R \#_{\xi} H, H, \pi, \sigma)$ associated to (R, ξ) as in Section 2.2.2, then (cf. [Scha, 6.1]) $\omega : R \#_{\xi} H \rightarrow A$ is a bialgebra isomorphism.

Conversely, we start with a pre-bialgebra with cocycle (R, ξ) in ${}^H_H\mathcal{YD}$ and construct the splitting datum $(R \#_{\xi} H, H, \pi, \sigma)$,

$$\sigma : H \hookrightarrow R \#_{\xi} H \quad \text{and} \quad \pi : R \#_{\xi} H \rightarrow H$$

as in Section 2.2.2. Then $(R \#_{\xi} H)^{co\pi} = R \# K$ and so the pre-bialgebra in ${}^H_H\mathcal{YD}$ associated to $(R \#_{\xi} H, H, \pi, \sigma)$ constructed in Section 2.2.3 is $R \otimes K$ which is isomorphic to R as a coalgebra in ${}^H_H\mathcal{YD}$ via the map $\theta : R \otimes K \rightarrow R$ where $\theta(r \otimes 1) = r$. The corresponding cocycle is ξ' where $\xi' = \xi(\theta \otimes \theta)$. Clearly θ induces an isomorphism of pre-bialgebras with cocycle between $(R \otimes K, \xi')$ and (R, ξ) .

In this situation we note that $\tau : (R \#_{\xi} H) \rightarrow (R \#_{\xi} H)^{co\pi}$ is given by $R \otimes \varepsilon$. For we have that

$$\begin{aligned} \tau(r \# h) &\stackrel{(10)}{=} (r^{(1)} \# r_{(-1)}^{(2)} h_{(1)}) \sigma S\pi(r_{(0)}^{(2)} \# h_{(2)}) \\ &= (r^{(1)} \# r_{(-1)}^{(2)} h_{(1)}) \sigma S(h_{(2)}) \varepsilon(r_{(0)}^{(2)}) \\ &= r \# \varepsilon(h). \end{aligned}$$

3. Associativity of (R, ξ)

In general, the multiplication in a pre-bialgebra R associated to a splitting datum (A, H, π, σ) need not be associative. It was noted in the previous section that m_R is associative if $\Psi(\xi)$ is the identity, or, equivalently, if $\xi(R \otimes R) \subset H$ acts trivially on R . First we consider some examples, and then show that the converse statement holds if R is connected.

Example 3.1. *Thin pre-bialgebras* A pre-bialgebra R is called *thin* if R is connected and the space of primitives of R is also one-dimensional. By [AMStu, Theorem 3.14], a finite-dimensional thin pre-bialgebra (R, ξ) has associative multiplication, even for nontrivial ξ , but need not be a bialgebra in ${}^H_H\mathcal{YD}$ by [AM, Example 6.4].

For Γ a finite abelian group, $V \in {}^\Gamma_\Gamma\mathcal{YD}$ (where we write ${}^\Gamma_\Gamma\mathcal{YD}$ for ${}^H_H\mathcal{YD}$ with $H = K[\Gamma]$), and $\mathcal{B}(V)$ the Nichols algebra of V , then all Hopf algebras whose associated graded Hopf algebra is $\mathcal{B}(V) \# K[\Gamma]$, called the liftings of $\mathcal{B}(V) \# K[\Gamma]$, are well known if V is a quantum linear space. Recall that for $g \in G$ and $\chi \in \widehat{\Gamma}$, V_g^χ is the set of $v \in V$ such that $\rho(v) = v_{(-1)} \otimes v_{(0)} = g \otimes v$ and $h \cdot v = \chi(h)v$ for all $h \in G$.

Definition 3.2. $V = \bigoplus_{i=1}^t K v_i \in {}^\Gamma_\Gamma\mathcal{YD}$ with $0 \neq v_i \in V_{g_i}^{\chi_i}$ with $g_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$ is called a quantum linear space when

$$\chi_i(g_j)\chi_j(g_i) = 1 \quad \text{for } i \neq j \text{ and } \chi_i(g_i) \text{ is a primitive } r_i\text{th root of } 1, \quad r_i > 1.$$

The following is proved in [AS2] or [BDG].

Proposition 3.3. For V a quantum linear space with $\chi_i(g_i)$ a primitive r_i th root of 1, all liftings $A := A(a_i, a_{ij} \mid 1 \leq i, j \leq t)$ of $\mathcal{B}(V) \# K[\Gamma]$ are Hopf algebras generated by the grouplikes and by $(1, g_i)$ -primitives x_i , $1 \leq i \leq t$ where

$$hx_i = \chi_i(h)x_ih,$$

$$x_i^{r_i} = a_i(1 - g_i^{r_i}) \quad \text{where } a_i = 0 \text{ if } g_i^{r_i} = 1 \text{ or } \chi_i^{r_i} \neq \varepsilon,$$

$$x_i x_j = \chi_j(g_i)x_j x_i + a_{ij}(1 - g_i g_j) \quad \text{where } a_{ij} = 0 \text{ if } g_i g_j = 1 \text{ or } \chi_i \chi_j \neq \varepsilon.$$

One sees directly from Proposition 3.3 that $a_{ji} = -\chi_j(g_i)^{-1}a_{ij} = -\chi_i(g_j)a_{ij}$. By rescaling, we may assume that the a_i are 0 or 1.

Example 3.4. Using the notation of Proposition 3.3, let $A := A(a_1, a_2, a_{12} = a)$ be a nontrivial lifting of $\mathcal{B}(V) \# K[\Gamma]$ where $V = Kx_1 \oplus Kx_2$ is a quantum plane as above. Then A has PBW basis $\{gx_1^i x_2^j \mid g \in \Gamma, 0 \leq i, j \leq r-1\}$, and the map $\pi : A \rightarrow H = K[\Gamma]$ defined by $\pi(gx_1^i x_2^j) = \delta_{0, i+j}g$ is an H -bilinear coalgebra homomorphism that splits the inclusion $H \xrightarrow{\sigma} A$. Thus (A, H, π, i) is a splitting datum and so $A \cong R \#_\xi H$ for some pre-bialgebra with cocycle (R, ξ) . Since $R = A^{co\pi}$ then R has K -basis $\{x_1^i x_2^j \mid 0 \leq i, j \leq r-1\}$. In general, (R, ξ) is not associative.

The next example shows that for $A = A(1, 1, a)$ as above, with $r > 2$, there is no choice of an H -bilinear projection π splitting the inclusion which will make the associated pre-bialgebra (R, ξ) associative.

Example 3.5. Let $A := A(1, 1, a)$ be the Hopf algebra described in Proposition 3.3 with $t = 2$, $a_1 = a_2 = 1$, $a_{12} = a \neq 0$ and $r := r_1 = r_2 > 2$. Then $\chi_1 = \chi_2^{-1}$, and $1 = \chi_1(g_2)\chi_2(g_1)$ implies that $\chi_1(g_1) =$

$\chi_2(g_2)^{-1}$. Let q denote $\chi_1(g_1)$. Then $x_2x_1 = qx_1x_2 + a(1 - g_1g_2)$. We show that there is no H -bilinear coalgebra morphism $\pi : A \rightarrow K[\Gamma] = H$ splitting the inclusion σ such that $R = A^{co\pi}$ is associative. The proof is by contradiction.

Suppose that π is such a morphism and $R = A^{co\pi}$ is associative. Then since g_1 is an invertible element, and

$$g_1\pi(x_1^n x_2^m) = \pi(g_1 x_1^n x_2^m) = q^{n-m}\pi(x_1^n x_2^m g_1) = q^{n-m}\pi(x_1^n x_2^m)g_1 = q^{n-m}g_1\pi(x_1^n x_2^m),$$

then

$$\pi(x_1^n x_2^m) = 0 \quad \text{if } n \neq m, \text{ and } 0 \leq n, m \leq r-1. \quad (18)$$

Next note that if $\pi : A \rightarrow H$ is as above, and if $\pi(x_1^i x_2^j) = 0$ for all $0 < i + j < u + v$, then $\pi(x_1^u x_2^v) = \beta_{u,v}(g_1^u g_2^v - 1)$, i.e., $\pi(x_1^u x_2^v)$ is $(1, g_1^u g_2^v)$ -primitive. For by the quantum binomial theorem [K], for scalars $\omega(i, j)$,

$$\Delta(x_1^u x_2^v) = g_1^u g_2^v \otimes x_1^u x_2^v + x_1^u x_2^v \otimes 1 + \sum_{\substack{0 \leq i \leq u, 0 \leq j \leq v \\ 0 < i+j < u+v}} \omega(i, j) g_1^i x_1^{u-i} g_2^j x_2^{v-j} \otimes x_1^i x_2^j$$

and applying $\pi \otimes \pi$ to this expression, we obtain that

$$\Delta(\pi(x_1^u x_2^v)) = g_1^u g_2^v \otimes \pi(x_1^u x_2^v) + \pi(x_1^u x_2^v) \otimes 1.$$

Using this argument with $0 < i + j = 1$ yields $\pi(x_1 x_2) = \beta(g_1 g_2 - 1)$ for some scalar β . We now test associativity on x_1, x_2, x_2 . First we have that

$$\begin{aligned} x_1 \cdot_R x_2 &= \tau(x_1 x_2) = g_1 g_2 \sigma(S_H(\pi(x_1 x_2))) + 0 + 0 + x_1 x_2 \\ &= \beta g_1 g_2 ((g_1 g_2)^{-1} - 1) + x_1 x_2 = x_1 x_2 - \beta(g_1 g_2 - 1), \end{aligned}$$

so that

$$(x_1 \cdot_R x_2) \cdot_R x_2 = \tau(x_1 x_2 x_2 - \beta(g_1 g_2 - 1)x_2) = \tau(x_1 x_2^2) - \beta(q^{-2} - 1)x_2.$$

On the other hand, since by (18), $\pi(x_2) = \pi(x_2^2) = 0$, then

$$x_2 \cdot_R x_2 = \tau(x_2^2) = x_2^2,$$

and thus

$$x_1 \cdot_R (x_2 \cdot_R x_2) = x_1 \cdot_R x_2^2 = \tau(x_1 x_2^2).$$

If R is associative then these expressions must be equal and thus $\beta = 0$. Now consider multiplication in R of the elements x_2, x_1, x_1 . First we compute

$$x_2 \cdot_R x_1 = \tau(x_2 x_1) = \tau(qx_1 x_2 + a(g_1 g_2 - 1)) = q\tau(x_1 x_2) = qx_1 x_2 \quad \text{since } \beta = 0.$$

Thus $(x_2 \cdot_R x_1) \cdot_R x_1 = \tau(qx_1x_2x_1)$. On the other hand

$$\begin{aligned} x_2 \cdot_R (x_1 \cdot_R x_1) &= x_2 \cdot_R x_1^2 = \tau(x_2x_1^2) \\ &= \tau(qx_1x_2x_1 + a(1 - g_1g_2)x_1) = \tau(qx_1x_2x_1) + a(q^2 - 1)x_1. \end{aligned}$$

This contradicts the choice of $a \neq 0$.

Remark 3.6. In the above example, it is key that $r > 2$ and $x_i^2 \neq 0$. In the examples of dimension 32 in Section 5, R is not thin, ξ is nontrivial, but R is a bialgebra in ${}^H_H\mathcal{YD}$ since the image of ξ lies in the centre of A .

We now prove the converse to the observation in Remark 2.11 in the case that R is connected.

Theorem 3.7. Let (R, ξ) be a pre-bialgebra with cocycle in ${}^H_H\mathcal{YD}$. If R is connected, then the following are equivalent.

- (i) m_R is associative.
- (ii) $\xi(z)t = \varepsilon(z)t$, for every $z \in R \otimes R$, $t \in R$.
- (iii) $\Phi(\xi) = \text{Id}_{R^{\otimes 3}}$.

Proof. By Remarks 2.8 and 2.11 it remains only to show that (i) implies (ii), i.e., to prove that if m_R is associative, then

$$\xi(r \otimes s)t = \varepsilon_R(r)\varepsilon_R(s)t, \quad \text{for every } r, s, t \in R. \quad (19)$$

The argument is by induction on $u + v$ where $r \in R_u$ and $s \in R_v$. For $u + v = 0, 1$, then either $u = 0$ or $v = 0$ and by (8), there is nothing to show.

Since R is connected, by [Mo, Lemma 5.3.2, 2)], for every $n > 0$ and $r \in R_n$ there exists a finite set I and $r_i, r^i \in R_{n-1}$, for every $i \in I$, such that

$$\Delta(r) = 1_R \otimes r + r \otimes 1_R + \sum_{i \in I} r_i \otimes r^i,$$

and thus

$$\sum_{i \in I} r_i \varepsilon_R(r^i) = -\varepsilon(r)1_R = \sum_{i \in I} \varepsilon_R(r_i)r^i. \quad (20)$$

Recall that by (11)

$$\begin{aligned} \Delta_A(r \# 1_H) &= r^{(1)} \# r_{(-1)}^{(2)} \otimes r_{(0)}^{(2)} \# 1_H \\ &= (1_R \# r_{(-1)} \otimes r_{(0)} \# 1_H) + (r \# 1_R \otimes 1_R \# 1_H) + \sum_{i \in I} (r_i \# r_{(-1)}^i \otimes r_{(0)}^i \# 1_H). \end{aligned}$$

Suppose that the statement holds for $u + v - 1$ and let $r \in R_u$ with comultiplication as above and $s \in R_v$ with $\Delta_R(s) = 1_R \otimes s + s \otimes 1_R + \sum_{j \in J} s_j \otimes s^j$. Let us compute $r \cdot_R s$. We have

$$\begin{aligned}
(r \# 1_H)(s \# 1_H) &\stackrel{(13)}{=} r^{(1)} \cdot_R (r_{(-1)}^{(2)} s^{(1)}) \# \xi(r_{(0)}^{(2)} \otimes s^{(2)}) \\
&= r_{(-1)} s^{(1)} \# \xi(r_{(0)} \otimes s^{(2)}) + r \cdot_R s^{(1)} \# \xi(1_R \otimes s^{(2)}) \\
&\quad + \sum_{i \in I} r_i \cdot_R (r_{(-1)}^i s^{(1)}) \# \xi(r_{(0)}^i \otimes s^{(2)}) \\
&= r \cdot_R s \# 1_H + r_{(-1)} s^{(1)} \# \xi(r_{(0)} \otimes s^{(2)}) \\
&\quad + \sum_{i \in I} r_i \cdot_R (r_{(-1)}^i s^{(1)}) \# \xi(r_{(0)}^i \otimes s^{(2)}) \\
&= r \cdot_R s \# 1_H + \left[\begin{array}{l} r_{(-1)} 1_R \# \xi(r_{(0)} \otimes s) \\ + r_{(-1)} s \# \xi(r_{(0)} \otimes 1_R) \\ + \sum_{j \in J} r_{(-1)} s_j \# \xi(r_{(0)} \otimes s^j) \end{array} \right] \\
&\quad + \left[\begin{array}{l} \sum_{i \in I} r_i \cdot_R (r_{(-1)}^i 1_R) \# \xi(r_{(0)}^i \otimes s) \\ + \sum_{i \in I} r_i \cdot_R (r_{(-1)}^i s) \# \xi(r_{(0)}^i \otimes 1_R) \\ + \sum_{i \in I} \sum_{j \in J} r_i \cdot_R (r_{(-1)}^i s_j) \# \xi(r_{(0)}^i \otimes s^j) \end{array} \right] \\
&= r \cdot_R s \# 1_H + \left[\begin{array}{l} 1_R \# \xi(r \otimes s) + \varepsilon_R(r) s \# 1_H \\ + \sum_{j \in J} r_{(-1)} s_j \# \xi(r_{(0)} \otimes s^j) \end{array} \right] \\
&\quad + \left[\begin{array}{l} \sum_{i \in I} r_i \# \xi(r^i \otimes s) \\ + \sum_{i \in I} r_i \varepsilon_R(r^i) \cdot_R s \# 1_H \\ + \sum_{i \in I} \sum_{j \in J} r_i \cdot_R (r_{(-1)}^i s_j) \# \xi(r_{(0)}^i \otimes s^j) \end{array} \right]
\end{aligned}$$

so that

$$(r \# 1_H)(s \# 1_H) = \left[\begin{array}{l} r \cdot_R s \# 1_H + 1_R \# \xi(r \otimes s) + \sum_{i \in I} r_i \# \xi(r^i \otimes s) \\ + \sum_{j \in J} r_{(-1)} s_j \# \xi(r_{(0)} \otimes s^j) + \sum_{i \in I} \sum_{j \in J} r_i \cdot_R (r_{(-1)}^i s_j) \# \xi(r_{(0)}^i \otimes s^j) \end{array} \right].$$

Note that $(R \otimes \varepsilon_H)[(r \# 1_H)(s \# h)] = r \cdot_R (R \otimes \varepsilon_H)(s \# h)$ so that

$$(R \otimes \varepsilon_H)[(r \# 1_H)(s \# 1_H)(t \# 1_H)] = r \cdot_R (R \otimes \varepsilon_H)[(s \# 1_H)(t \# 1_H)] = r \cdot_R (s \cdot_R t).$$

Then, we have

$$\begin{aligned}
0 &= r \cdot_R (s \cdot_R t) - (r \cdot_R s) \cdot_R t \\
&= (R \otimes \varepsilon_H)[(r \# 1_H)(s \# 1_H)(t \# 1_H)] - (R \otimes \varepsilon_H)[(r \cdot_R s \# 1_H)(t \# 1_H)] \\
&= (R \otimes \varepsilon_H)\left\{[(r \# 1_H)(s \# 1_H) - r \cdot_R s \# 1_H](t \# 1_H)\right\} \\
&= (R \otimes \varepsilon_H) \left[\begin{array}{l} (1_R \# \xi(r \otimes s))(t \# 1_H) \\ + \sum_{i \in I} [r_i \# \xi(r^i \otimes s)](t \# 1_H) \\ + \sum_{j \in J} [r_{(-1)} s_j \# \xi(r_{(0)} \otimes s^j)](t \# 1_H) \\ + \sum_{i \in I} \sum_{j \in J} [r_i \cdot_R (r_{(-1)}^i s_j) \# \xi(r_{(0)}^i \otimes s^j)](t \# 1_H) \end{array} \right].
\end{aligned}$$

The first term in this sum is clearly $\xi(r \otimes s)t$ and it remains to show that the other terms add to $-\varepsilon_R(r)\varepsilon_R(s)t$.

Since $r^i \in R_{u-1}$, the second term in the sum above is

$$\begin{aligned} (R \otimes \varepsilon_H) & \left[\sum_{i \in I} (r_i \# \xi(r^i \otimes s))(t \# 1) \right] \\ & \stackrel{(15)}{=} \sum_{i \in I} r_i \cdot_R (\xi(r^i \# s)t) \\ & = \sum_{i \in I} r_i \cdot_R (\varepsilon_R(r^i)\varepsilon_R(s)t) \text{ by the induction hypothesis} \\ & \stackrel{(19)}{=} -\varepsilon_R(r)\varepsilon_R(s)t. \end{aligned}$$

A similar computation shows that $(R \otimes \varepsilon_H)(\sum_{j \in J} [r_{(-1)}s_j \# \xi(r_{(0)} \otimes s^j)](t \# 1_H)) = -\varepsilon_R(r)\varepsilon_R(s)t$ and that $(R \otimes \varepsilon_H)(\sum_{j \in J} [r_i \cdot_R (r_{(-1)}^i s_j) \# \xi(r_{(0)}^i \otimes s^j)](t \# 1_H)) = \varepsilon_R(r)\varepsilon_R(s)t$ so that

$$0 = \xi(r \otimes s)t - \varepsilon_R(r)\varepsilon_H(s)t - \varepsilon_R(r)\varepsilon_R(s)t + \varepsilon_H(r)\varepsilon_H(s)t = \xi(r \otimes s)t - \varepsilon_R(r)\varepsilon_H(s)t$$

and the proof is finished. \square

4. Cocycle deformations of splitting data

Let (A, H, π, σ) be a splitting datum with associated pre-bialgebra with cocycle (R, ξ) . In this section, we extend the notion of a cocycle deformation of A to a cocycle deformation of R and show how these are related. For Γ a finite abelian group, V a crossed $k[\Gamma]$ module and $A = \mathcal{B}(V) \# K[\Gamma]$, then the results we present should be compared to those in [GM, Section 4]

Recall that if A is a bialgebra, a convolution invertible map $\gamma : A \otimes A \rightarrow K$ is called a unital (or normalized) 2-cocycle for A when for all $x, y, z \in A$,

$$\gamma(y_{(1)} \otimes z_{(1)})\gamma(x \otimes y_{(2)}z_{(2)}) = \gamma(x_{(1)} \otimes y_{(1)})\gamma(x_{(2)}y_{(2)} \otimes z), \quad (21)$$

$$\gamma(x \otimes 1) = \gamma(1 \otimes x) = \varepsilon_A(x). \quad (22)$$

Note that (21) holds for all $x, y, z \in A$ if and only if

$$(\varepsilon_A \otimes \gamma) * \gamma(A \otimes m_A) = (\gamma \otimes \varepsilon_A) * \gamma(m_A \otimes A).$$

For a bialgebra A with a sub-Hopf algebra H , we denote by $Z_H^2(A, K)$ the space of H -bilinear 2-cocycles for A , i.e., the set of cocycles as defined above which are also H -bilinear. If $H = K$ we write $Z^2(A, K)$ instead of $Z_H^2(A, K)$.

One may deform or twist A by any $\gamma \in Z^2(A, K)$ to get a new bialgebra A^γ . (See, for example, [Doi, Theorem 1.6].) As a coalgebra, $A^\gamma = A$, but the multiplication in A^γ is given by

$$x \cdot_\gamma y = x \cdot_{A^\gamma} y := \gamma(x_{(1)} \otimes y_{(1)})x_{(2)}y_{(2)}\gamma^{-1}(x_{(3)} \otimes y_{(3)}),$$

for all $x, y \in A$. By (22), A^γ has unit 1_A and condition (21) implies that the multiplication in A^γ is associative if and only if the multiplication in A is associative. If A is a Hopf algebra with antipode S , by [Doi, 1.6(a5)] then $\gamma(x_{(1)} \otimes S(x_{(2)}))\gamma^{-1}(S(x_{(3)}) \otimes x_{(4)}) = \varepsilon(x)$, so that A^γ is also a Hopf algebra

with antipode given by

$$S^\gamma(x) = \gamma(x_{(1)} \otimes S(x_{(2)}))S(x_{(3)})\gamma^{-1}(S(x_{(4)}) \otimes x_{(5)}).$$

By [Doi, 1.6(a3)], γ^{-1} is a cocycle for A^{cop} and, as algebras, $A^\gamma = (A^{\text{cop}})^{\gamma^{-1}}$.

Definition 4.1. Let A be a bialgebra and let $\beta, \gamma : A \otimes A \rightarrow K$ be K -bilinear maps. Denote by ${}_\beta A_\gamma$ the vector space A endowed with the following not necessarily associative multiplication

$$x \cdot {}_\beta A_\gamma y = \beta(x_{(1)} \otimes y_{(1)})x_{(2)}y_{(2)}\gamma(x_{(3)} \otimes y_{(3)}), \quad \text{for all } x, y \in A.$$

Remark 4.2. For A, γ, β as above, if $\gamma = \varepsilon_{A \otimes A}$, then we denote ${}_\beta A_\gamma$ simply by ${}_\beta A$. The multiplication of ${}_\beta A$ is just denoted by $*_\beta$ where $x *_\beta y = \beta(x_{(1)} \otimes y_{(1)})x_{(2)}y_{(2)}$. Then β satisfies (21) and (22) if and only if $(A, *_\beta)$ is an associative algebra with $1_A = 1_{(A, *_\beta)}$. The condition on 1_A is equivalent to (22). The associativity statement follows from computing

$$\begin{aligned} (x *_\beta y) *_\beta z &= \beta(x_{(1)} \otimes y_{(1)})\beta(x_{(2)}y_{(2)} \otimes z_{(1)})x_{(3)}y_{(3)}z_{(2)} \\ &= [(\beta \otimes \varepsilon_A) * \beta(m_A \otimes A)](x_{(1)} \otimes y_{(1)} \otimes z_{(1)})x_{(2)}y_{(2)}z_{(2)} \end{aligned}$$

and

$$\begin{aligned} x *_\beta (y *_\beta z) &= \beta(y_{(1)} \otimes z_{(1)})\beta(x_{(1)} \otimes y_{(2)}z_{(2)})x_{(2)}y_{(3)}z_{(3)} \\ &= [(\varepsilon_A \otimes \beta) * \beta(A \otimes m_A)](x_{(1)} \otimes y_{(1)} \otimes z_{(1)})x_{(2)}y_{(2)}z_{(2)}. \end{aligned}$$

Thus clearly if β satisfies (21), then $*_\beta$ is an associative operation, and, applying ε to the expressions above, we see that the converse holds.

Similarly, if $\beta = \varepsilon_{A \otimes A}$, we denote $A_\gamma := {}_\beta A_\gamma$. The multiplication of A_γ will be simply denoted by $*^\gamma$ so that it is defined by $x *^\gamma y = x_{(1)}y_{(1)}\gamma(x_{(2)} \otimes y_{(2)})$. Then $*^\gamma$ is an associative operation if and only if γ satisfies (21) for A^{cop} . Then if A is a bialgebra, γ satisfies (21) and (22) for A^{cop} if and only if $A_\gamma := (A, *^\gamma)$ is associative with unit 1_A .

Observe that, for $\gamma \in Z^2(A, K)$, one has $A^\gamma = {}_\gamma A_{\gamma^{-1}}$ as an algebra.

The next lemma will be useful in building examples in the last section of this paper.

Lemma 4.3. For A a bialgebra, let $\beta, \gamma : A \otimes A \rightarrow K$ be K -bilinear convolution invertible maps. Suppose that ${}_\beta A_\gamma$ is an associative unitary algebra. Then $\beta \in Z^2(A, K)$ if and only if $\gamma^{-1} \in Z^2(A, K)$.

Proof. For any K -bilinear map $\sigma : A \otimes A \rightarrow K$, we define maps $X(\sigma), Y(\sigma) : A \otimes A \otimes A \rightarrow K$ by

$$\begin{aligned} X(\sigma)(a \otimes b \otimes c) &:= \sigma(a_{(1)} \otimes b_{(1)})\sigma(a_{(2)}b_{(2)} \otimes c), \\ Y(\sigma)(a \otimes b \otimes c) &:= \sigma(b_{(1)} \otimes c_{(1)})\sigma(a \otimes b_{(2)}c_{(2)}), \end{aligned}$$

for all $a, b, c \in A$. Thus σ satisfies (21) if and only if $X(\sigma) = Y(\sigma)$. We have

$$\begin{aligned} (a \cdot {}_\beta A_\gamma b) \cdot {}_\beta A_\gamma c &= (\beta(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)}\gamma(a_{(3)} \otimes b_{(3)})) \cdot {}_\beta A_\gamma c \\ &= \beta(a_{(1)} \otimes b_{(1)})\beta(a_{(2)}b_{(2)} \otimes c_{(1)})a_{(3)}b_{(3)}c_{(2)}\gamma(a_{(4)}b_{(4)} \otimes c_{(3)})\gamma(a_{(5)} \otimes b_{(5)}) \\ &= X(\beta)(a_{(1)} \otimes b_{(1)} \otimes c_{(1)})a_{(2)}b_{(2)}c_{(2)}[X(\gamma^{-1})]^{-1}(a_{(3)} \otimes b_{(3)} \otimes c_{(3)}), \end{aligned}$$

and

$$\begin{aligned} a \cdot_{\beta A_\gamma} (b \cdot_{\beta A_\gamma} c) &= a \cdot_{\beta A_\gamma} (\beta(b_{(1)} \otimes c_{(1)})b_{(2)}c_{(2)}\gamma(b_{(3)} \otimes c_{(3)})) \\ &= \beta(b_{(1)} \otimes c_{(1)})\beta(a_{(1)} \otimes b_{(2)}c_{(2)})a_{(2)}b_{(3)}c_{(3)}\gamma(a_{(3)} \otimes b_{(4)}c_{(4)})\gamma(b_{(5)} \otimes c_{(5)}) \\ &= Y(\beta)(a_{(1)} \otimes b_{(1)} \otimes c_{(1)})a_{(2)}b_{(2)}c_{(2)}[Y(\gamma^{-1})]^{-1}(a_{(3)} \otimes b_{(3)} \otimes c_{(3)}) \end{aligned}$$

where $[X(\gamma^{-1})]^{-1}$ and $[Y(\gamma^{-1})]^{-1}$ denote the convolution inverses of $X(\gamma^{-1})$ and $Y(\gamma^{-1})$ respectively. Since $(a \cdot_{\beta A_\gamma} b) \cdot_{\beta A_\gamma} c = a \cdot_{\beta A_\gamma} (b \cdot_{\beta A_\gamma} c)$, by applying ε_A to both sides we obtain

$$X(\beta) * [X(\gamma^{-1})]^{-1} = Y(\beta) * [Y(\gamma^{-1})]^{-1}$$

that is,

$$[X(\beta)]^{-1} * Y(\beta) = [X(\gamma^{-1})]^{-1} * Y(\gamma^{-1}).$$

It is now clear that β satisfies (21) if and only if γ^{-1} does.

We have

$$b = 1 \cdot_{\beta A_\gamma} b = \beta(1_{(1)} \otimes b_{(1)})1_{(2)}b_{(2)}\gamma(1_{(3)} \otimes b_{(3)})$$

so that

$$b = \beta(1 \otimes b_{(1)})b_{(2)}\gamma(1 \otimes b_{(3)}).$$

By applying ε_A to both sides, we obtain $\varepsilon_A(b) = \beta(1 \otimes b_{(1)})\gamma(1 \otimes b_{(2)})$ which yields

$$\beta(1 \otimes -) = \gamma^{-1}(1 \otimes -).$$

Similarly $a = a \cdot_{\beta A_\gamma} 1$ yields $\beta(- \otimes 1) = \gamma^{-1}(- \otimes 1)$. Therefore β satisfies (22) if and only if γ^{-1} does. \square

Corollary 4.4. For A a bialgebra, let $\beta \in Z^2(A, K)$ and $\gamma \in Z^2(A^\beta, K)$. Then $\gamma * \beta \in Z^2(A, K)$.

Proof. By Remark 4.2, $\gamma(A^\beta)$ is associative. Now $\gamma(A^\beta) = \gamma * \beta A_{\beta^{-1}}$ so that, by Lemma 4.3, $\gamma * \beta \in Z^2(A, K)$. \square

A map $\gamma \in \text{Hom}(A \otimes A, K)$ is called H -balanced if $\gamma : A \otimes_H A \rightarrow K$, in other words, for all $a, a' \in A$, $h \in H$,

$$\gamma(a\sigma(h) \otimes a') = \gamma(a \otimes \sigma(h)a'). \quad (23)$$

Lemma 4.5. Let A be a bialgebra, H a Hopf algebra and $\sigma : H \rightarrow A$ a bialgebra monomorphism. Let $\gamma \in Z_H^2(A, K)$. Then

- (i) γ is H -balanced.
- (ii) γ^{-1} is also H -bilinear and H -balanced.

Proof. (i) By applying (21) with $y = \sigma(h)$ and using H -bilinearity of γ , we get that γ is H -balanced.
(ii) For $a, a' \in A, h, h' \in H$, we have

$$\begin{aligned} & \gamma^{-1}(\sigma(h)a \otimes a'\sigma(h')) \\ &= \gamma^{-1}(\sigma(h)a_{(1)} \otimes a'_{(1)}\sigma(h'))\gamma(a_{(2)} \otimes a'_{(2)})\gamma^{-1}(a_{(3)} \otimes a'_{(3)}) \\ &= \gamma^{-1}(\sigma(h)_{(1)}a_{(1)} \otimes a'_{(1)}\sigma(h')_{(1)})\gamma(\sigma(h)_{(2)}a_{(2)} \otimes a'_{(2)}\sigma(h')_{(2)})\gamma^{-1}(a_{(3)} \otimes a'_{(3)}) \\ &= \varepsilon_H(h)\gamma^{-1}(a \otimes a')\varepsilon_H(h'), \end{aligned}$$

and so γ^{-1} is H -bilinear. Similarly, write $\gamma^{-1}(a\sigma(h) \otimes a')$ as $\gamma^{-1}(a_1\sigma(h_1) \otimes a'_1)\gamma(a_2 \otimes \sigma(h_2)a'_2)\gamma^{-1} \times (a_3 \otimes \sigma(h_3)a'_3)$, to see that γ^{-1} is H -balanced. \square

Lemma 4.6. Let (A, H, π, σ) be a splitting datum and let $\gamma \in Z_H^2(A, K)$. Then $(A^\gamma, H, \pi, \sigma)$ is also a splitting datum with $A^{\text{co}\pi} = A^{\gamma\text{co}\pi}$ as coalgebras in ${}^H_H\mathcal{YD}$.

Proof. Since $A^\gamma = A$ as coalgebras, in order to prove that $(A^\gamma, H, \pi, \sigma)$ is a splitting datum we have to check that σ is an algebra homomorphism and that π is H -bilinear. Since both γ and γ^{-1} are H -bilinear, for $h, h' \in H$ and $a \in A$, we get

$$\begin{aligned} \sigma(h) \cdot_\gamma a &= \gamma(\sigma(h_{(1)}) \otimes a_{(1)})\sigma(h_{(2)})a_{(2)}\gamma^{-1}(\sigma(h_{(3)}) \otimes a_{(3)}) \\ &= \gamma(1_A \otimes a_{(1)})\sigma(h)a_{(2)}\gamma^{-1}(1_A \otimes a_{(3)}) = \sigma(h)a. \end{aligned}$$

Similarly $a \cdot_\gamma \sigma(h) = a\sigma(h)$. Thus $\sigma(h) \cdot_\gamma \sigma(h') = \sigma(h)\sigma(h') = \sigma(hh')$ and $\pi(\sigma(h) \cdot_\gamma a \cdot_\gamma \sigma(h')) = \pi(\sigma(h)a\sigma(h')) = h\pi(a)h'$ for all $h, h' \in H$ and $a \in A$. Hence $(A^\gamma, H, \pi, \sigma)$ is a splitting datum. The corresponding map $\tau_\gamma : A^\gamma \rightarrow R$, as in 2.2.3 is given by

$$\tau_\gamma(a) = a_{(1)} \cdot_\gamma \sigma S\pi(a_{(2)}) = a_{(1)}\sigma S\pi(a_{(2)}) = \tau(a).$$

Using this fact, the last part of the statement follows by definition of the coalgebra structures of $A^{\text{co}\pi}$ and $A^{\gamma\text{co}\pi}$ in ${}^H_H\mathcal{YD}$ as given in 2.2.3. \square

Now we offer an appropriate definition for a 2-cocycle $\nu : R \otimes R \rightarrow K$.

Definition 4.7. A convolution invertible map $\nu : R \otimes R \rightarrow K$ (where $R \otimes R$ has the coalgebra structure in 2.1) is called a unital 2-cocycle for (R, ξ) if for $\Phi(\xi) \in \text{End}(R \otimes R \otimes R)$ from Lemma 2.7,

$$(\varepsilon_R \otimes \nu) * \nu(R \otimes m_R) = (\nu \otimes \varepsilon_R) * \{\nu(m_R \otimes R)\Phi(\xi)\} \quad \text{and} \quad (24)$$

$$\nu(- \otimes 1_H) = \varepsilon_H = \nu(1_H \otimes -). \quad (25)$$

We will denote by $Z_H^2(R, K)$ the space of left H -linear 2-cocycles for R .

Given $\nu \in Z_H^2(R, K)$, let R^ν be the coalgebra $R \in {}^H_H\mathcal{YD}$ with multiplication defined by

$$m_{R^\nu} := (\nu \otimes m_R \otimes \nu^{-1})\Delta_{R \otimes R}^2 \quad \text{and unit} \quad u_{R^\nu} = u_R.$$

We will see in Theorem 4.11 that R^ν is also a pre-bialgebra with cocycle.

Lemma 4.8. Let (R, ξ) be a pre-bialgebra with cocycle and $(A = R \#_{\xi} H, H, \pi, \sigma)$ be the associated splitting datum. Let $\phi : A \otimes A \rightarrow K$ be H -bilinear and H -balanced. Then for $r, s, t \in R, h \in H$, the following hold:

$$\phi[r \# 1_H \otimes (s \# 1_H)(t \# 1_H)] = \phi(r \# 1_H \otimes st \# 1_H), \quad (26)$$

$$\phi(r \# h \otimes s \# 1_H) = \phi(r \# 1_H \otimes hs \# 1_H), \quad (27)$$

$$\phi(hr \# 1_H \otimes s \# 1_H) = \phi(r \# 1_H \otimes S(h)s \# 1_H). \quad (28)$$

Proof. The first statement holds since, using right H -linearity at the second step,

$$\begin{aligned} \phi[r \# 1_H \otimes (s \# 1_H)(t \# 1_H)] &= \phi[r \# 1_H \otimes (m_R \otimes \xi) \Delta_{R \otimes R}(S \otimes t)] \\ &= \phi[r \# 1_H \otimes (m_R \otimes u_H \varepsilon_H \xi) \Delta_{R \otimes R}(S \otimes t)] \\ &= \phi(r \# 1_H \otimes st \# 1_H). \end{aligned}$$

The second equation follows from:

$$\begin{aligned} \phi(r \# h \otimes s \# 1_H) &= \phi[(r \# 1_H)(1_R \# h) \otimes s \# 1_H] \\ &= \phi[r \# 1_H \otimes (1_R \# h)(s \# 1_H)] \quad (\phi \text{ } H\text{-balanced}) \\ &= \phi(r \# 1_H \otimes h_{(1)}s \# h_{(2)}) \\ &= \phi[r \# 1_H \otimes (h_{(1)}s \# 1_H)(1_R \# h_{(2)})] \\ &= \phi(r \# 1_H \otimes h_{(1)}s \# 1_H) \varepsilon_H(h_{(2)}) \quad (\phi \text{ } H\text{-bilinear}) \\ &= \phi(r \# 1_H \otimes hs \# 1_H). \end{aligned}$$

Finally we check (28),

$$\begin{aligned} \phi(hr \# 1_H \otimes s \# 1_H) &= \phi[(1_R \# h_{(1)})(r \# S(h_{(2)})) \otimes s \# 1_H] \\ &= \varepsilon_H(h_{(1)}) \phi[(r \# S(h_{(2)})) \otimes s \# 1_H] \quad (H\text{-balanced}) \\ &= \phi(r \# S(h) \otimes s \# 1_H) \\ &\stackrel{(27)}{=} \phi(r \# 1_H \otimes S(h)s \# 1_H). \quad \square \end{aligned}$$

Now let $\mathcal{BB}(A)$ denote the set of H -bilinear H -balanced maps from $A \otimes A$ to K and $\mathcal{L}(R)$ the set of left H -linear maps from $R \otimes R$ to K . The next proposition sets the stage for our first theorem.

Proposition 4.9. Let (R, ξ) be a pre-bialgebra with cocycle and $(A = R \#_{\xi} H, H, \pi, \sigma)$ be the associated splitting datum. There is a bijective correspondence between $\mathcal{BB}(A)$ and $\mathcal{L}(R)$ given by:

$$\begin{aligned} \Omega : \text{Hom}(A \otimes A, K) &\rightarrow \text{Hom}(R \otimes R, K) \quad \text{by } \gamma \mapsto \gamma_R := \gamma|_{R \otimes R} \text{ with inverse} \\ \Omega' : \text{Hom}(R \otimes R, K) &\rightarrow \text{Hom}(A \otimes A, K) \quad \text{by } \nu \mapsto \nu_A := \nu \circ (R \otimes \mu \otimes \varepsilon_H), \\ \text{so that } \nu_A(x \# h \otimes y \# h') &= \nu(x \otimes hy) \varepsilon_H(h') \quad \text{for each } h \in H, r, s \in R. \end{aligned}$$

Furthermore $\mathcal{BB}(A)$ and $\mathcal{L}(R)$ are both closed under the convolution product and Ω and Ω' preserve convolution.

Proof. Let $\gamma \in \mathcal{BB}(A)$ and we wish to show that $\Omega(\gamma) = \gamma_R$ is in $\mathcal{L}(R)$. By (28), we have $\gamma_R(hr \otimes s) = \gamma_R(r \otimes S(h)s)$ and thus

$$\gamma_R(h_{(1)}r \otimes h_{(2)}s) = \gamma_R(r \otimes S(h_{(1)})h_{(2)}s) = \varepsilon_H(h)\gamma_R(r \otimes s),$$

and γ_R is left H -linear. Conversely suppose $\nu \in \mathcal{L}(R)$ and check that $\Omega'(\nu) = \nu_A$ is H -bilinear. For $h, h', l, m \in H$ and $x, y \in R$,

$$\begin{aligned} \nu_A[(1_R \# l)(x \# h) \otimes (y \# h')(1_R \# m)] &= \nu_A(l_{(1)}x \# l_{(2)}h \otimes y \# h'm) \\ &\stackrel{\text{defn}}{=} \nu(l_{(1)}x \otimes l_{(2)}hy)\varepsilon_H(h'm) \\ &= \varepsilon_H(l)\nu(x \otimes hy)\varepsilon_H(h'm) \\ &= \varepsilon_H(l)\nu_A(x \# h \otimes y \# h')\varepsilon_H(m). \end{aligned}$$

The fact that ν_A is H -balanced follows directly from the definition.

For $r, s \in R$ and $h, m \in H$, we have that for $\gamma \in \mathcal{BB}(A)$,

$$[\Omega'(\gamma_R)](r \# h \otimes s \# m) = \gamma_R(r \otimes hs)\varepsilon_H(m) = \gamma(r \# 1 \otimes hs \# m) = \gamma(r \# h \otimes s \# m),$$

and for $\nu \in \mathcal{L}(R)$,

$$[\Omega(\nu_A)](r \otimes s) = \nu_A(r \# 1 \otimes s \# 1) = \nu(r \otimes s).$$

Thus Ω and Ω' are inverse bijections. For $\gamma, \gamma' \in \mathcal{BB}(A)$, it is clear that $\gamma * \gamma'$ is H -bilinear and H -balanced. Also for $\nu, \nu' \in \mathcal{L}(R)$, $h \in H$, $r, s \in R$,

$$\begin{aligned} (\nu * \nu')(h_{(1)}r \otimes h_{(2)}s) &= \nu(h_{(1)}r^{(1)} \otimes (h_{(2)}r^{(2)})_{(-1)}h_{(3)}s^{(1)})\nu'((h_{(2)}r^{(2)})_{(0)} \otimes h_{(4)}s^{(2)}) \\ &= \nu(h_{(1)}r^{(1)} \otimes h_{(2)}r_{(-1)}^{(2)}S(h_{(4)})h_{(5)}s^{(1)})\nu'(h_{(3)}r_{(0)}^{(2)} \otimes h_{(6)}s^{(2)}) \\ &= \varepsilon_H(h)\nu(r^{(1)} \otimes r_{(-1)}^{(2)}s^{(1)})\nu'(r_{(0)}^{(2)} \otimes s^{(2)}) \\ &= \varepsilon_H(h)(\nu * \nu')(r \otimes s). \end{aligned}$$

Thus $\mathcal{BB}(A)$ and $\mathcal{L}(R)$ are closed under convolution and it remains to show that Ω, Ω' are convolution preserving. First we let $\gamma, \gamma' \in \mathcal{BB}(A)$ and we check that $\gamma_R * \gamma'_R = (\gamma * \gamma')_R$. For every $x, y \in R$, we have

$$\begin{aligned} (\gamma_R * \gamma'_R)(x \otimes y) &= \gamma_R[(x \otimes y)^{(1)}]\gamma'_R[(x \otimes y)^{(2)}] \\ &= \gamma_R(x^{(1)} \otimes x_{(-1)}^{(2)}y^{(1)})\gamma'_R(x_{(0)}^{(2)} \otimes y^{(2)}) \\ &= \gamma(x^{(1)} \# 1_H \otimes x_{(-1)}^{(2)}y^{(1)} \# 1_H)\gamma'(x_{(0)}^{(2)} \# 1_H \otimes y^{(2)} \# 1_H) \\ &\stackrel{(27)}{=} \gamma[x^{(1)} \# x_{(-1)}^{(2)} \otimes y^{(1)} \# 1_H]\gamma'(x_{(0)}^{(2)} \# 1_H \otimes y^{(2)} \# 1_H) \\ &= \gamma[x^{(1)} \# x_{(-1)}^{(2)} \otimes y^{(1)} \# y_{(-1)}^{(2)}]\gamma'(x_{(0)}^{(2)} \# 1_H \otimes y_{(0)}^{(2)} \# 1_H) \\ &= \gamma[(x \# 1_H)_{(1)} \otimes (y \# 1_H)_{(1)}]\gamma'((x \# 1_H)_{(2)} \otimes (y \# 1_H)_{(2)}) \\ &= (\gamma * \gamma')(x \# 1_H \otimes y \# 1_H) = (\gamma * \gamma')_R(x \otimes y). \end{aligned}$$

Finally, to see that $\Omega'(\nu * \nu') = \Omega'(\nu) * \Omega'(\nu')$, apply Ω to both sides and use the fact that Ω is one-one and convolution preserving. \square

In fact, Ω maps 2-cocycles to 2-cocycles.

Theorem 4.10. *Let (R, ξ) be a pre-bialgebra with cocycle with $(A = R \#_{\xi} H, H, \pi, \sigma)$ the associated splitting datum. Then Ω and Ω' as above are inverse bijections between $Z_H^2(A, K)$ and $Z_H^2(R, K)$.*

Proof. First we note that clearly Ω, Ω' preserve the normality conditions (22) and (25). It remains to show that Ω, Ω' are compatible with the cocycle conditions (21) and (24).

Let $\nu \in Z_H^2(R, K)$. We will show that for $x, y, z \in R, h, h', h'' \in H$, then the left (right) hand side of (21) for $\gamma = \nu_A$ acting on $x \# h \otimes y \# h' \otimes z \# h''$ equals the left (right) hand side of (24) applied to $(x \otimes h_1 y \otimes h_2 h' z) \varepsilon(h'')$. Thus ν_A satisfies (21) if and only if ν satisfies (24). (To see the “if” implication, let $1 = h = h' = h''$.) We start with the left-hand side of (21) with $\gamma = \nu_A$,

$$\begin{aligned} & (\varepsilon_A \otimes \nu_A) * \nu_A(A \otimes m_A)(x \# h \otimes y \# h' \otimes z \# h'') \\ &= \nu_A((y \# h')_{(1)} \otimes (z \# h'')_{(1)}) \nu_A(x \# h \otimes (y \# h')_{(2)} (z \# h'')_{(2)}) \\ &\stackrel{(10)}{=} \nu_A(y^{(1)} \# y_{(-1)}^{(2)} h'_{(1)} \otimes z^{(1)} \# z_{(-1)}^{(2)} h''_{(1)}) \nu_A(x \# h \otimes (y_{(0)}^{(2)} \# h'_{(2)}) (z_{(0)}^{(2)} \# h''_{(2)})). \end{aligned} \quad (29)$$

By (13) and the right H -linearity of ν_A , we have

$$\nu_A(x \# h \otimes (y_{(0)}^{(2)} \# h'_{(2)}) (z_{(0)}^{(2)} \# h''_{(2)})) = \nu_A(x \# h \otimes y_{(0)}^{(2)} (h'_{(2)} z_{(0)}^{(2)}) \# 1_H) \varepsilon_H(h''_{(2)}).$$

Thus, from the definition of Ω' , expression (29) is equal to

$$\begin{aligned} & \nu(y^{(1)} \otimes y_{(-1)}^{(2)} h'_{(1)} z^{(1)}) \varepsilon_H(z_{(-1)}^{(2)} h''_{(1)}) \nu(x \otimes h(y_{(0)}^{(2)} (h'_{(2)} z_{(0)}^{(2)}))) \varepsilon_H(h''_{(2)}) \\ &= \nu(y^{(1)} \otimes y_{(-1)}^{(2)} h'_{(1)} z^{(1)}) \nu(x \otimes h(y_{(0)}^{(2)} (h'_{(2)} z_{(0)}^{(2)}))) \varepsilon_H(h''). \end{aligned}$$

Then we use left H -linearity to obtain that

$$\begin{aligned} & \nu(y^{(1)} \otimes y_{(-1)}^{(2)} h'_{(1)} z^{(1)}) \nu(x \otimes h(y_{(0)}^{(2)} (h'_{(2)} z_{(0)}^{(2)}))) \\ &= \nu(x_{(-2)} h_{(1)} y^{(1)} \otimes x_{(-1)} h_{(2)} y_{(-1)}^{(2)} S(h_{(4)}) (h_{(5)} h' z)^{(1)}) \nu(x_{(0)} \otimes (h_{(3)} y_{(0)}^{(2)}) ((h_{(5)} h' z)^{(2)})) \\ &= \nu(x_{(-2)} (h_{(1)} y)^{(1)} \otimes x_{(-1)} (h_{(1)} y)_{(-1)}^{(2)} (h_{(2)} h' z)^{(1)}) \nu(x_{(0)} \otimes (h_{(1)} y)_{(0)}^{(2)} (h_{(2)} h' z)^{(2)}) \end{aligned}$$

and thus we have that (29) equals

$$[(\varepsilon_R \otimes \nu) * \nu(R \otimes m_R)](x \otimes h_{(1)} y \otimes h_{(2)} h' z) \varepsilon_H(h'')$$

as claimed. Now we tackle the right-hand side of (21) for $\gamma = \nu_A$,

$$\begin{aligned}
& (\nu_A \otimes \varepsilon_A) * \nu_A(m_A \otimes A)(x \# h \otimes y \# h' \otimes z \# h'') \\
&= \nu_A((x \# h)_{(1)} \otimes (y \# h')_{(1)}) \nu_A[(x \# h)_{(2)}(y \# h')_{(2)} \otimes z \# h''] \\
&\stackrel{(10)}{=} \nu_A(x^{(1)} \# x_{(-1)}^{(2)} h_{(1)} \otimes y^{(1)} \# y_{(-1)}^{(2)} h'_{(1)}) \nu_A[(x_{(0)}^{(2)} \# h_{(2)})(y_{(0)}^{(2)} \# h'_{(2)}) \otimes z \# h''] \\
&= \nu(x^{(1)} \otimes x_{(-1)}^{(2)} h_{(1)} y^{(1)}) \varepsilon_H(y_{(-1)}^{(2)} h'_{(1)}) \nu_A[(x_{(0)}^{(2)} \# h_{(2)})(y_{(0)}^{(2)} \# h'_{(2)}) \otimes z \# h''] \\
&= \nu(x^{(1)} \otimes x_{(-1)}^{(2)} h_{(1)} y^{(1)}) \nu_A[(x_{(0)}^{(2)} \# h_{(2)})(y^{(2)} \# h') \otimes z \# h''].
\end{aligned}$$

But by (13),

$$\begin{aligned}
(x_{(0)}^{(2)} \# h_{(2)})(y^{(2)} \# h') &= (R \otimes m_H)[(m_R \otimes \xi) \Delta_{R \otimes R} \otimes H](x_{(0)}^{(2)} \otimes h_{(2)} y^{(2)} \otimes h_{(3)} h') \\
&= (x_{(0)}^{(2)})^{(1)} ((x_{(0)}^{(2)})_{(-1)}^{(2)} h_{(2)} y^{(2)}) \# \xi[(x_{(0)}^{(2)})_{(0)}^{(2)} \otimes h_{(3)} y^{(3)}] h_{(4)} h',
\end{aligned}$$

and

$$\begin{aligned}
& \nu_A[(x_{(0)}^{(2)})^{(1)} ((x_{(0)}^{(2)})_{(-1)}^{(2)} h_{(2)} y^{(2)}) \# \xi[(x_{(0)}^{(2)})_{(0)}^{(2)} \otimes h_{(3)} y^{(3)}] h_{(4)} h' \otimes z \# h''] \\
&= \nu[(x_{(0)}^{(2)})^{(1)} ((x_{(0)}^{(2)})_{(-1)}^{(2)} h_{(2)} y^{(2)}) \otimes \xi[(x_{(0)}^{(2)})_{(0)}^{(2)} \otimes h_{(3)} y^{(3)}] h_{(4)} h' z] \varepsilon_H(h'').
\end{aligned}$$

Thus $\nu_A(A \otimes A \otimes \varepsilon_A) * \nu_A(m_A \otimes A)(x \# h \otimes y \# h' \otimes z \# h'')$ is exactly

$$[(\nu \otimes \varepsilon_R) * \{ \nu(m_R \otimes R) \Phi(\xi) \}](x \otimes h_{(1)} y \otimes h_{(2)} h' z) \varepsilon_H(h'').$$

This proves the theorem. \square

Recall from Lemma 4.6 that if (A, H, π, σ) is a splitting datum with associated pre-bialgebra with cocycle (R, ξ) , and if $\gamma \in Z_H^2(A, K)$, then $(A^\gamma, H, \pi, \sigma)$ is also a splitting datum and has associated pre-bialgebra with cocycle (R, η) for some η . The next theorem describes this relationship more precisely.

Theorem 4.11. *Let (R, ξ) be a pre-bialgebra with cocycle, and let $\gamma \in Z_H^2(R \#_\xi H, K)$. Define $\xi_{\gamma_R} : R^{\gamma_R} \otimes R^{\gamma_R} \rightarrow H$ by*

$$\xi_{\gamma_R} = u_H \gamma_R * \xi * \Psi(\gamma_R^{-1}) = u_H \gamma_R * \xi * (H \otimes \gamma_R^{-1}) \rho_{R \otimes R}.$$

Let $(A := R \#_\xi H, H, \pi, \sigma)$ be the splitting datum of 2.2.2 so that by 2.2.4, the associated pre-bialgebra with cocycle is $(R \otimes K, \xi(\theta \otimes \theta))$, where $\theta : R \otimes K \rightarrow R$ is the usual isomorphism. Then $(A^\gamma = (R \#_\xi H)^\gamma, H, \pi, \sigma)$ is also a splitting datum whose associated pre-bialgebra with cocycle is $(R \otimes K, \xi_{\gamma_R}(\theta \otimes \theta))$. Furthermore $(R^\gamma, \xi_{\gamma_R})$ is a pre-bialgebra with cocycle isomorphic to $(R \otimes K, \xi_{\gamma_R})$ via θ and

$$A^\gamma = (R \#_\xi H)^\gamma = R^{\gamma_R} \#_{\xi_{\gamma_R}} H.$$

Proof. By Lemma 2.3, $\Psi(\gamma_R^{-1}) = (H \otimes \gamma_R^{-1}) \rho_{R \otimes R}$ is convolution invertible with inverse $\Psi(\gamma_R)$. For $A := R \#_\xi H$ with associated pre-bialgebra with cocycle $(R \otimes K, \xi)$, let $(Q = R \otimes K, \zeta)$ denote the pre-bialgebra with cocycle associated to $(A^\gamma, H, \pi, \sigma)$. For $x \otimes 1, y \otimes 1 \in Q$, since $\tau = R \otimes \varepsilon_H$ from Section 2.2.4, multiplication in Q is given by

$$\begin{aligned}
& (x \otimes 1_K) \cdot_Q (y \otimes 1_K) \\
&= \tau[(x \# 1_H) \cdot_{A^\vee} (y \# 1_H)] \\
&= (R \otimes \varepsilon_H)[(x \# 1_H) \cdot_{A^\vee} (y \# 1_H)] \\
&\stackrel{(15)}{=} \gamma_R(x^{(1)} \otimes x_{(-1)}^{(2)} x_{(-3)}^{(3)} y^{(1)})(x_{(0)}^{(2)}) \cdot_R (x_{(-1)}^{(3)} y^{(2)}) \otimes \varepsilon_H(y_{(-1)}^{(3)}) \gamma_R^{-1}(x_{(0)}^{(3)} \otimes y_{(0)}^{(3)}) \\
&= \gamma_R(x^{(1)} \otimes x_{(-1)}^{(2)} x_{(-2)}^{(3)} y^{(1)} x_{(0)}^{(2)}) \cdot_R (x_{(-1)}^{(3)} y^{(2)}) \gamma_R^{-1}(x_{(0)}^{(3)} \otimes y_{(0)}^{(3)}) \otimes 1_K \\
&= [(\gamma_R \otimes m_R \otimes \gamma_R^{-1}) \Delta_{R \otimes R}^2(x \otimes y)] \otimes 1_K \\
&= m_{R^{\vee R}}(x \otimes y) \otimes 1_K.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\zeta(x \otimes 1_K \otimes y \otimes 1_K) &= \pi[(x \# 1_H) \cdot_{A^\vee} (y \# 1_H)] \\
&= (\varepsilon_R \otimes H)[(x \# 1_H) \cdot_{A^\vee} (y \# 1_H)] \\
&\stackrel{(16)}{=} \gamma_R(x^{(1)} \otimes x_{(-1)}^{(2)} x_{(-3)}^{(3)} y^{(1)}) \xi(x_{(0)}^{(2)} \otimes x_{(-2)}^{(3)} y^{(2)}) x_{(-1)}^{(3)} y_{(-1)}^{(3)} \gamma_R^{-1}(x_{(0)}^{(3)} \otimes y_{(0)}^{(3)}) \\
&= \gamma_R(x^{(1)} \otimes x_{(-1)}^{(2)} x_{(-2)}^{(3)} y^{(1)}) \xi(x_{(0)}^{(2)} \otimes x_{(-1)}^{(3)} y^{(2)}) (H \otimes \gamma_R^{-1}) \rho_{R \otimes R}(x_{(0)}^{(3)} \otimes y_{(0)}^{(3)}) \\
&= [u_H \gamma_R * \xi * (H \otimes \gamma_R^{-1}) \rho_{R \otimes R}](x \otimes y) \\
&= \xi_{\gamma_R}(x \otimes y).
\end{aligned}$$

Thus θ , the isomorphism of coalgebras in ${}^H_H\mathcal{YD}$ from Section 2.2.4, induces the structure of a pre-bialgebra with cocycle on the image $\theta(Q)$ and the image is exactly $R^{\vee R}$. Moreover $(R^{\vee R}, \xi_{\gamma_R})$ is indeed a pre-bialgebra with cocycle as claimed, and $\zeta = \xi_{\gamma_R}$.

As in Section 2.2.3, we can consider the bialgebra isomorphism

$$\omega^{-1} : A^\vee \rightarrow Q \#_\zeta H, \quad \omega^{-1}(a) = \tau(a_{(1)}) \otimes \pi(a_{(2)}).$$

We have

$$\begin{aligned}
\omega^{-1}(r \otimes h) &= \tau(r^{(1)} \# r_{(-1)}^{(2)} h_{(1)}) \otimes \pi(r_{(0)}^{(2)} \# h_{(2)}) \\
&= (r^{(1)} \otimes \varepsilon_H(r_{(-1)}^{(2)} h_{(1)})) \otimes \varepsilon_R(r_{(0)}^{(2)} h_{(2)}) \\
&= r \otimes 1_K \otimes h.
\end{aligned}$$

Since $\theta : (Q, \zeta) \rightarrow (R^\vee, \xi_{\gamma_R})$ is an isomorphism of pre-bialgebras with cocycle (see Section 2.2.1) we get that

$$\theta \otimes H : Q \#_\zeta H \rightarrow R^{\vee R} \#_{\xi_{\gamma_R}} H$$

is a bialgebra isomorphism. Therefore $(\theta \otimes H)\omega^{-1} : A^\vee \rightarrow R^{\vee R} \#_{\xi_{\gamma_R}} H$ is a bialgebra isomorphism too. By the foregoing $\text{Id}_{R \otimes H} = (\theta \otimes H)\omega^{-1}$. \square

Corollary 4.12. *Let (A, H, π, σ) be a splitting datum, let (R, ξ) be the associated pre-bialgebra with cocycle, and let $\gamma \in Z_H^2(A, K)$. Then $(R^{\vee R}, \xi_{\gamma_R})$ is exactly the pre-bialgebra with cocycle associated to the splitting datum (A^\vee, H, π, σ) .*

Proof. As in Section 2.2.3, we can consider the bialgebra isomorphisms

$$\omega : R \#_{\xi} H \rightarrow A, \quad \omega(r \otimes h) = r \cdot_A \sigma(h).$$

This gives a bialgebra isomorphism

$$\omega^{\gamma} : (R \#_{\xi} H)^{\gamma(\omega \otimes \omega)} \rightarrow A^{\gamma}, \quad \omega^{\gamma}(r \otimes h) = \omega(r \otimes h).$$

Let $\alpha := \gamma(\omega \otimes \omega)$. By Theorem 4.11, we have

$$(R \#_{\xi} H)^{\alpha} = R^{\alpha_R} \#_{\xi_{\alpha_R}} H.$$

We have

$$\alpha_R(r \otimes s) = \alpha(r \otimes 1_H \otimes s \otimes 1_H) = \gamma(\omega \otimes \omega)(r \otimes 1_H \otimes s \otimes 1_H) = \gamma(r \otimes s) = \gamma_R(r \otimes s)$$

so that $\alpha_R = \gamma_R$ whence

$$(R \#_{\xi} H)^{\alpha} = R^{\gamma_R} \#_{\xi_{\gamma_R}} H.$$

Denote by (Q, ζ) the pre-bialgebra associated to $(A^{\gamma}, H, \pi, \sigma)$. We have

$$Q = (A^{\gamma})^{coH} = (A)^{coH} = R$$

so that $Q \#_{\zeta} H = R \otimes H$ as a vector space. The isomorphism corresponding to the splitting datum $(A^{\gamma}, H, \pi, \sigma)$ is given by

$$\omega' : Q \#_{\zeta} H \rightarrow A^{\gamma}, \quad \omega'(r \otimes h) = r \cdot_{A^{\gamma}} \sigma(h).$$

We have

$$\begin{aligned} r \cdot_{A^{\gamma}} \sigma(h) &= \gamma(r_{(1)} \otimes \sigma(h)_{(1)}) r_{(2)} \cdot_A \sigma(h)_{(2)} \gamma^{-1}(r_{(3)} \otimes \sigma(h)_{(3)}) \\ &= \gamma(r_{(1)} \otimes \sigma(h_{(1)})) r_{(2)} \cdot_A \sigma(h_{(2)}) \gamma^{-1}(r_{(3)} \otimes \sigma(h_{(3)})) \\ &= \gamma(r_{(1)} \sigma(h_{(1)}) \otimes 1_A) r_{(2)} \cdot_A \sigma(h_{(2)}) \gamma^{-1}(r_{(3)} \sigma(h_{(3)}) \otimes 1_A) \\ &= r \cdot_A \sigma(h) \end{aligned}$$

so that $\omega' = \omega^{\gamma}$. Hence we get that

$$\text{Id}_{R \otimes H} = (\omega')^{-1} \omega^{\gamma} : R^{\gamma_R} \#_{\xi_{\gamma_R}} H \rightarrow Q \#_{\zeta} H$$

is a bialgebra isomorphism. Now $(R^{\gamma_R} \#_{\xi_{\gamma_R}} H, H, \pi', \sigma')$ and $(Q \#_{\zeta} H, H, \pi', \sigma')$ are both splitting data where $\pi' : R \otimes H \rightarrow H$, $\pi'(r \otimes h) = \varepsilon_A(r)h$ and $\sigma' : H \rightarrow R \otimes H$, $\sigma'(h) = 1_R \otimes h$. Therefore, in view of [AM, Proposition 1.15], one gets that $(R^{\gamma_R}, \xi_{\gamma_R}) = (Q, \zeta)$ as pre-bialgebras with cocycle. \square

We consider when the conditions which ensure associativity of R also hold for a cocycle twist R^{γ} .

Corollary 4.13. Let $A = R \#_{\xi} H$, $\gamma \in Z_H^2(A, K)$, and $A^\gamma = R^u \#_{\xi_u} H$ as in Theorem 4.11 where $u := \Omega(\gamma) = \gamma_R$ and $\xi_u := \xi_{\gamma_R}$. By Theorem 3.7, we have

- (a) $\xi(z)t = \varepsilon(z)t$, for every $z \in R \otimes R$, $t \in R$ if and only if $\Phi(\xi) = \text{Id}_{R^{\otimes 3}}$, and,
 (b) $\xi_u(z)t = \varepsilon(z)t$, for every $z \in R \otimes R$, $t \in R$ if and only if $\Phi(\xi_u) = \text{Id}_{R^{\otimes 3}}$.

If $\Phi(\Psi(u)) = \Phi(u_H u)$ then (a) and (b) are equivalent. Conversely, if (a) and (b) both hold, then $\Phi(\Psi(u)) = \Phi(u_H u)$.

Proof. Suppose first that $\Phi(\Psi(u)) = \Phi(u_H u)$. Since by Theorem 4.11, $\Phi(\xi_u) = \Phi(u_H u * \xi * \Psi(u^{-1}))$, and by Lemma 2.7, Φ is an algebra map, then clearly $\Phi(\xi_u)$ is the identity if and only if $\Phi(\xi)$ is. Conversely if $\Phi(\xi_u) = \text{Id}_{R^{\otimes 3}} = \Phi(\xi)$, then by Theorem 4.11, $\Phi(u_H u * \Psi(u^{-1})) = \text{Id}_{R^{\otimes 3}}$. \square

5. Cocycle twists for Radford biproducts of quantum planes

5.1. Construction of the cocycle

Let Γ be a finite abelian group, let $H = K[\Gamma]$ and let $V = Kx_1 \oplus Kx_2 \in {}_{\Gamma}^{\Gamma} \mathcal{VD}$ be a quantum plane with $x_i \in V_{g_i}^{x_i}$ as in Definition 3.2. Let A be the Radford biproduct $\mathcal{B}(V) \# H$. Suppose as well that $g_1 g_2 \neq 1$, $\chi_1 \chi_2 = \varepsilon$, $g_i^r \neq 1$ and $\chi_i^r = \varepsilon$ so that it makes sense for the scalars a_i and a to be nonzero.

Let $\chi := \chi_1 = \chi_2^{-1}$. Suppose that $\chi(g_1) = q$ is a primitive r th root of unity. Then $\chi_1(g_1) = \chi_1(g_2) = q$ and $\chi_2(g_1) = \chi_2(g_2) = q^{-1}$.

Although it is known that liftings of the coradically graded Hopf algebra A are isomorphic to cocycle twists of A , the explicit description of the cocycle in the most general setting has not been given. In this section we will describe this cocycle.

Lemma 5.1. Let $\gamma \in \text{Hom}(A \otimes A, K)$ be H -bilinear and H -balanced. If $q^{i+k} \neq q^{j+l}$ then $\gamma(x_1^i x_2^j \otimes x_1^k x_2^l) = 0$.

Proof. Suppose $\gamma(x_1^i x_2^j \otimes x_1^k x_2^l) \neq 0$. Since γ is H -balanced, $\gamma(x_1^i x_2^j g_1 \otimes x_1^k x_2^l) = \gamma(x_1^i x_2^j \otimes g_1 x_1^k x_2^l)$ so that from the H -bilinearity of γ then $\chi^{j-i}(g_1) = \chi^{k-l}(g_1)$. Thus $q^{j-i} = q^{k-l}$ and the result follows. \square

Corollary 5.2. Let $\gamma \in Z_H^2(A, K)$. If $q^{i+k} \neq q^{j+l}$, then $\gamma^{\pm 1}(x_1^i x_2^j \otimes x_1^k x_2^l) = 0$.

Proof. By Lemma 4.5, γ is H -balanced, and γ^{-1} is H -bilinear and H -balanced also. \square

The next propositions will require the q -analogue of the Chu–Vandermonde formula [K, Proposition IV.2.3]

$$\sum_{k=0}^r \binom{a}{k}_q \binom{b}{r-k}_q q^{(a-k)(r-k)} = \binom{a+b}{r}_q, \quad (30)$$

as well as the fact that when q is a primitive r th root of unity and $k \leq r$,

$$\binom{r+k}{k}_q = 1. \quad (31)$$

Also we will need the fact, which follows directly from the q -binomial theorem, that

$$\binom{n}{k}_{q^{-1}} q^{k(n-k)} = \binom{n}{k}_q. \quad (32)$$

If n , i or $n-i$ is negative, we set $\binom{n}{i}_q = 0$.

Proposition 5.3. Let $A = R \# H$ as above with $R = \mathcal{B}(V)$, $H = K[\Gamma]$. Define H -bilinear maps γ_i , $i = 1, 2$, from $A \otimes A$ to K as follows: $\gamma_i = \varepsilon$ on all $x_1^i x_2^m \otimes x_1^n x_2^t$ except that

$$\gamma_i(x_i^m \otimes x_i^{r-m}) = a_i, \quad (33)$$

and γ_i is then extended to $A \otimes A$ by H -bilinearity.

The maps γ_1, γ_2 lie in $Z_H^2(A, K)$ and furthermore these cocycles commute.

Proof. We show first that $\gamma_1 \in Z_H^2(A, K)$. Note that by the definition, γ_i is H -balanced. By the definition of γ_i , condition (22) holds and we check condition (21) for the triple $x_1^j x_2^l, x_1^k x_2^t, x_1^s x_2^r$. It is clear from the definition of γ_1 that both sides of (21) are 0 unless $j = l = s = 0$ so that the triple to check is x_1^j, x_1^k, x_1^t . By Lemma 5.1, both sides are 0 unless $i + k + t = r$ or $i + k + t = 2r$. Suppose first that $i + k + t = r$. Then the left-hand side equals $\gamma_1(g_1^k \otimes g_1^t) \gamma_1(x_1^i \otimes x_1^k x_1^t) = a_1$ and similarly the right-hand side is $\gamma_1(x_1^i x_1^k \otimes x_1^t) = a_1$.

Now let $i + k + t = 2r$. Then the left-hand side of (21) is

$$\begin{aligned} & \sum_{m \geq 0} \binom{k}{m}_{q^{-1}} \binom{t}{r-m}_q \gamma_1(x_1^m \otimes x_1^{r-m}) \gamma_1(x_1^i \otimes x_1^{k-m} x_1^{t-(r-m)}) \\ & \stackrel{(32)}{=} \left(\sum_{m \geq 0} q^{-m(k-m)} \binom{k}{m}_q \binom{t}{r-m}_q \right) a_1^2 \\ & \stackrel{(30)}{=} \binom{k+t}{r}_q a_1^2 \stackrel{(31)}{=} a_1^2. \end{aligned}$$

Similarly the right-hand side of (21) is

$$\sum_{n \geq 0} \binom{i}{n}_{q^{-1}} \binom{k}{r-n}_q \gamma_1(x_1^n \otimes x_1^{r-n}) \gamma_1(x_1^{i-n} x_1^{k-(r-n)} \otimes x_1^t) = \binom{i+k}{r}_q a_1^2 = a_1^2.$$

The proof that γ_2 is a cocycle is analogous.

We show next that γ_1 and γ_2 commute by applying $\gamma_1 * \gamma_2$ and $\gamma_2 * \gamma_1$ to $x_1^i x_2^j \otimes x_1^k x_2^l$. Then

$$\gamma_1 * \gamma_2(x_1^i x_2^j \otimes x_1^k x_2^l) = \gamma_1(x_1^i g_2^j \otimes x_1^k) \gamma_2(x_2^j \otimes x_2^l) = q^{-ij} \delta_{i+k, r} \delta_{j+l, r} a_1 a_2,$$

while

$$\gamma_2 * \gamma_1(x_1^i x_2^j \otimes x_1^k x_2^l) = \gamma_2(x_2^j \otimes g_1^k x_2^l) \gamma_1(x_1^i \otimes x_1^k) = q^{-kl} \delta_{i+k, r} \delta_{j+l, r} a_1 a_2.$$

Since for $i + k = j + l = r$ then $q^{-ij} = q^{-(r-k)(r-l)} = q^{-kl}$, these expressions are equal. \square

Note that it is straightforward to check that γ_i^{-1} is the H -bilinear map defined exactly as γ_i is but with a_i replaced by $-a_i$. Also note that the multiplication $m_i : A^{\gamma_i} \otimes A^{\gamma_i} \rightarrow A^{\gamma_i}$ is the same as the multiplication on A except that for $0 < m < r$,

$$m_i(x_i^m \otimes x_i^{r-m}) = \gamma_i(x_i^m \otimes x_i^{r-m}) + x_i^r + g_i^r \gamma_i^{-1}(x_i^m \otimes x_i^{r-m}) = a_i(1 - g_i^r).$$

Corollary 5.4. For $i \neq j$, $i, j = 1, 2$, $\gamma_i \in Z_H^2(A^{\gamma_j}, K)$.

Proof. Basically the same proof as that of Proposition 5.3 shows that $\gamma_i \in Z_H^2(A^{\gamma_i}, K)$, $i \neq j$. For example, to show that $\gamma_2 \in Z_H^2(A^{\gamma_1}, K)$, we test the triple $x_1^i x_2^j, x_1^k x_2^l, x_1^t x_2^s$ and find that the left-hand side of (21) is

$$\begin{aligned} & \sum_m \binom{l}{m}_q \binom{s}{r-m}_{q^{-1}} \gamma_2(g_1^k x_2^m \otimes g_1^t x_2^{r-m}) \gamma_2(x_1^i x_2^j \otimes x_1^{k-l-m} x_2^t x_2^{s-r+m}) \\ &= \sum_m \binom{l}{m}_q \binom{s}{r-m}_{q^{-1}} q^{(l-m)t} \gamma_2(x_2^m \otimes g_1^k x_2^{r-m}) \gamma_2(x_1^i x_2^j \otimes x_1^{k+t} x_2^{l-m} x_2^{s-r+m}). \end{aligned}$$

Clearly this is 0 unless $i = 0$. If $0 < k + t \neq r$, then this expression is also clearly 0. If $k + t = r$ then $x_1^{k+t} = a_1(1 - g_1^r)$ and since g_1^r commutes with x_2 and γ_2 is H -bilinear, we have 0 here too. Thus the left-hand side is 0 unless $t = k = i = 0$ and the right-hand side computation is similar. Thus the computation simplifies to that in Proposition 5.3. \square

Corollary 5.5. For $i, j = 1, 2$ and $i \neq j$, then $\gamma_i * \gamma_j \in Z_H^2(A, K)$.

Proof. By Proposition 5.3, $\gamma_j \in Z_H^2(A, K)$ and by Corollary 5.4 we have $\gamma_i \in Z_H^2(A^{\gamma_j}, K)$. The statement then follows from Corollary 4.4. \square

Note that the multiplication m' in $A^{\gamma_1 * \gamma_2}$ is the same as that of A except that for $0 < l, m < r$,

$$m'(x_1^l x_2^m \otimes x_1^{r-l} x_2^{r-m}) = q^{-lm} a_1 a_2 (1 - g_1^r)(1 - g_2^r).$$

Now we consider a cocycle which twists the multiplication of x_1 and x_2 .

Proposition 5.6. Let $A = R \# H$ as above with $R = \mathcal{B}(V)$, $H = K[\Gamma]$. Define the H -bilinear map γ_a from $A \otimes A$ to K as follows: $\gamma_a = \varepsilon$ on all $x_1^l x_2^m \otimes x_1^n x_2^t$ except that

$$\gamma_a(x_2^m \otimes x_1^m) = (m)!_q a^m, \quad (34)$$

and γ_a is then extended to all of $A \otimes A$ by H -bilinearity. Then $\gamma_a \in Z_H^2(A^{\gamma_1 * \gamma_2}, K)$.

Proof. Let β denote $\gamma_1 * \gamma_2 = \gamma_2 * \gamma_1$. We check that $\gamma_a \in Z_H^2(A^\beta, K)$ by applying the left and right-hand sides of Eq. (21) to the triple $x_1^i x_2^j, x_1^k x_2^l, x_1^t x_2^s$.

The left-hand side is equal to:

$$\begin{aligned} & \sum_m \binom{l}{m}_q \binom{t}{m}_q \gamma_a(g_1^k x_2^m \otimes x_1^m g_2^s) \gamma_a(x_1^i x_2^j \otimes x_1^{k-l-m} x_2^t x_1^{t-m} x_2^s) \\ &= \delta_{i,0} \sum_m \binom{l}{m}_q \binom{t}{m}_q (m)!_q a^m q^{(l-m)(t-m)} \gamma_a(x_2^j \otimes x_1^k x_1^{t-m} x_2^{l-m} x_2^s). \end{aligned}$$

This expression is 0 unless $s = 0$ and $l = m$. If $l - m + s \neq r$ this is clear, and if $l - m + s = r$, then we have $\gamma_a(x_2^j \otimes x_1^k x_1^{t-m} (a_2(1 - g_2^r)))$ which is 0 by the H -bilinearity of γ_a . Thus:

$$lhs = \delta_{i,0} \delta_{s,0} \delta_{k+t,l+j} \binom{t}{l}_q (l)!_q (j)!_q a^{j+l}.$$

Similarly the right-hand side equals:

$$\delta_{i,0}\delta_{s,0} \binom{j}{k}_q \gamma_a(x_2^k \otimes x_1^k) \gamma_a(x_2^{j-k} x_2^l \otimes x_1^t) = \delta_{i,0}\delta_{s,0} \binom{j}{k}_q \delta_{j-k+l,t} (k)_q! (t)_q! q^{k+t},$$

and it is an easy exercise to see that $\binom{j}{l}_q (l)_q! (j)_q! = \binom{j}{k}_q (k)_q! (t)_q! q$. Thus $\gamma_a \in Z_H^2(A^\beta, K)$. \square

We note that the cocycles γ_i and γ_a do not commute. For example, consider

$$\gamma_1 * \gamma_a(x_1^{r-1} x_2 \otimes x_1^2) = \binom{2}{1}_q \gamma_1(x_1^{r-1} g_2 \otimes x_1) \gamma_a(x_2 \otimes x_1) = q \binom{2}{1}_q a_1 a,$$

while

$$\gamma_a * \gamma_1(x_1^{r-1} x_2 \otimes x_1^2) = \binom{2}{1}_q \gamma_a(g_1^{r-1} x_2 \otimes x_1) \gamma_1(x_1^{r-1} \otimes x_1) = \binom{2}{1}_q a_1 a.$$

Similar examples show that γ_2 and γ_a do not commute.

Corollary 5.7. $\gamma_a \in Z_H^2(A, K)$ and $\gamma_a \in Z_H^2(A^{\gamma_i}, K)$ for $i = 1, 2$.

Proof. Note that the a_i are any scalars. If $a_i = 0$ then $A^{\gamma_i * \gamma_j} = A^{\gamma_j}$ and if $a_1 = a_2 = 0$ then $A^{\gamma_i * \gamma_j} = A$. \square

Note that the cocycles $\gamma_a \in Z_H^2(A, K)$ and γ_i (for a quantum line) were described in [GM, Section 5.3] in terms of Hochschild cohomology.

Corollary 5.8. $\alpha := \gamma_a * \gamma_1 * \gamma_2 \in Z_H^2(A, K)$.

Proof. By Corollary 5.5, $\gamma_1 * \gamma_2 \in Z_H^2(A, K)$ and, by Proposition 5.6, $\gamma_a \in Z_H^2(A^{\gamma_1 * \gamma_2}, K)$. The statement then follows from Corollary 4.4. \square

We now describe the cocycle twist of A^α of A . We will need the fact that $\gamma_a^{-1}(x_2 \otimes x_1) = -a$; this is easy to check.

Proposition 5.9. Let $\alpha = \gamma_a * \gamma_1 * \gamma_2 \in Z_H^2(A, K)$. Then A^α is isomorphic to the lifting $A(a_1, a_2, a)$ of A described in Proposition 3.3.

Proof. We must describe the multiplication \cdot_α in the Hopf algebra A^α . Note that $x_i^n \cdot_\alpha x_i^m = x_i^{n+m}$ for $n + m < r$ since each of the cocycles γ_i and γ_a , and their inverses, are 0 on $x_i^k \otimes x_i^l$ when $j + l < r$. If $n + m = r$ then

$$x_i^n \cdot_\alpha x_i^m = \alpha(g_i^n \otimes g_i^m) x_i^r \alpha^{-1}(1 \otimes 1) + \alpha(x_i^n \otimes x_i^m) + g_i^r \alpha^{-1}(x_i^n \otimes x_i^m) = a_i(1 - g_i^r).$$

Now note that by the definition of α , then $x_1^n \cdot_\alpha x_2^m = x_1^n x_2^m$. However

$$x_2 \cdot_\alpha x_1 = \alpha(x_2 \otimes x_1) + x_2 x_1 + g_2 g_1 \alpha^{-1}(x_2 \otimes x_1) = q x_1 \cdot_\alpha x_2 + a(1 - g_2 g_1)$$

as required. Since multiplication is associative, this completes the proof. \square

We summarize the action of the cocycle α on $A \otimes A$. For $0 < i, k, m, n, t < r$ we have

- (i) $\alpha(z \otimes 1) = \alpha(1 \otimes z) = \varepsilon(z)$ for all $z \in A$.
- (ii) $\alpha(x_i^n \otimes x_i^m) = \delta_{n+m,r} a_i$.
- (iii) $\alpha(x_1^m \otimes x_2^k) = 0$.
- (iv) $\alpha(x_1^i \otimes x_1^k x_2^m) = 0 = \alpha(x_1^i x_2^k \otimes x_2^m)$.
- (v) $\alpha(x_2^m \otimes x_1^n) = \delta_{n,m}(m)!_q a^m$.
- (vi) $\alpha(x_2^i \otimes x_1^k x_2^m) = \delta_{i+m,r+k} \binom{i}{k}_q (k)!_q a^k a_2$.
- (vii) $\alpha(x_1^i x_2^k \otimes x_1^m) = \delta_{i+m,r+k} \binom{m}{k}_q (k)!_q a^k a_1$.
- (viii) $\alpha(x_1^i x_2^k \otimes x_1^m x_2^t) = \delta_{i+m,k+t} (i+m-r)!_q \binom{k}{r-t}_q \binom{m}{r-i}_q q^{it} a^{i+m-r} a_1 a_2$.

Example 5.10. Let us describe α completely for the Hopf algebras of dimension 81 which were among the first counterexamples to Kaplansky's Tenth Conjecture. Here $\Gamma = \langle c \rangle$ is the cyclic group of order 9, $g_1 = c = g_2$, $\chi(c) = q$ where q is a primitive cube root of 1, $r = 3$. By [Mas2], there exists a cocycle α such that $A(a_1, a_2, a) \cong A^\alpha$. Here we supply γ explicitly for a_i and a nonzero.

From the preceding computations, we see that $\alpha = \varepsilon$ except for the following cases:

$$\begin{aligned}
 \alpha(x_2 \otimes x_1) &= (1)!_q a^1 = a, \\
 \alpha(x_i \otimes x_i^2) &= \alpha(x_i^2 \otimes x_i) = a_i \quad \text{for } i = 1, 2, \\
 \alpha(x_2^2 \otimes x_1^2) &= (2)!_q a^2 = (1+q)a^2, \\
 \alpha(x_2^2 \otimes x_1 x_2^2) &= \binom{2}{1}_q (1)!_q a a_2 = (2)!_q a a_2 = (1+q)a a_2, \\
 \alpha(x_1^2 x_2 \otimes x_1^2) &= \binom{2}{1}_q (1)!_q a a_1 = (2)!_q a a_1 = (1+q)a a_1, \\
 \alpha(x_1^2 x_2^2 \otimes x_1 x_2) &= (0)!_q \binom{2}{2}_q \binom{1}{1}_q q^2 a^0 a_1 a_2 = q^2 a_1 a_2 = -(1+q)a_1 a_2, \\
 \alpha(x_1^2 x_2 \otimes x_1 x_2^2) &= (0)!_q \binom{1}{1}_q \binom{1}{1}_q q^4 a^0 a_1 a_2 = q a_1 a_2, \\
 \alpha(x_1 x_2^2 \otimes x_1^2 x_2) &= (0)!_q \binom{2}{2}_q \binom{2}{2}_q q^1 a^0 a_1 a_2 = q a_1 a_2, \\
 \alpha(x_1 x_2 \otimes x_1^2 x_2^2) &= (0)!_q \binom{1}{1}_q \binom{2}{2}_q q^2 a^0 a_1 a_2 = q^2 a_1 a_2 = -(1+q)a_1 a_2, \\
 \alpha(x_1^2 x_2^2 \otimes x_1^2 x_2^2) &= (1)!_q \binom{2}{1}_q \binom{2}{1}_q q^4 a a_1 a_2 = (2)!_q (2)!_q q a a_1 a_2 \\
 &= (1+q)^2 q a a_1 a_2 = -(1+q)a a_1 a_2.
 \end{aligned}$$

In the last case, $i+m=k+t=4$, and in the four preceding cases, $i+m=k+t=r=3$.

We ask whether, in the example above, it is possible to find η such that $\alpha = e^\eta = \varepsilon + \eta + \frac{\eta^2}{2!} + \frac{\eta^3}{3!} + \dots$. This is the approach of [GM] to construct cocycles where it may be simpler to construct η . Then one expects that

$$\eta = \ln \alpha = \ln(\varepsilon + (\alpha - \varepsilon)) = (\alpha - \varepsilon) - \frac{(\alpha - \varepsilon)^2}{2} + \frac{(\alpha - \varepsilon)^3}{3} - \dots$$

and one checks (using a computer algebra system) that $(\alpha - \varepsilon)^3 = 0$ so that $\eta = \ln \alpha = \ln(\varepsilon + (\alpha - \varepsilon)) = (\alpha - \varepsilon) - \frac{(\alpha - \varepsilon)^2}{2}$. The map η is explicitly given by the table:

$$\eta = \begin{bmatrix} \eta(u \otimes v) & v=1 & x_1 & x_2 & x_1x_2 & x_1^2 & x_2^2 & x_1x_2^2 & x_1^2x_2 & x_1^2x_2^2 \\ u=1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 \\ x_2 & 0 & a & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\ x_1x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1^2 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_2^2 & 0 & 0 & a_2 & 0 & (1+\frac{q}{2})a^2 & 0 & (\frac{1}{2}+q)aa_2 & 0 & 0 \\ x_1x_2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1^2x_2 & 0 & 0 & 0 & 0 & (\frac{1}{2}+q)aa_1 & 0 & 0 & 0 & 0 \\ x_1^2x_2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}aa_1a_2 \end{bmatrix}$$

and $e^\eta = \alpha$.

5.2. Pointed Hopf algebras of dimension 32

In the next example, we study three infinite families of pointed Hopf algebras of dimension 32. Let Γ be a finite abelian group of order 8 and let $V \in {}_{\Gamma}^{\Gamma}\mathcal{VD}$ be a quantum linear space as in Definition 3.2 with $r_i = 2$, $i = 1, 2$. For each of the Γ, V listed below, the families of pointed Hopf algebras obtained by lifting the Radford biproduct yield infinite families of non-isomorphic Hopf algebras [G, Section 5]; family (F2) is also mentioned in [B]. We have shown by the above explicit computation of the cocycle that the pointed Hopf algebras in each of the families of liftings are isomorphic to a twisting of the Radford biproduct. The same result, without an explicit formula for the cocycle, can also be found as follows. In view of [EG, Theorem 3.1] each element in a family (F_i) is of the form $A(G_i, V_i, u_i, B)^*$ for some datum (G_i, V_i, u_i, B) . The construction of the Hopf algebra $A(G_i, V_i, u_i, B)$ can be found at the beginning of [EG, Section 2]: it can be obtained by applying [AEG, Theorem 3.1.1] to the datum $((\mathbb{C}[G_i] \times \wedge V_i)^{e^B}, u_i)$. Now, by construction, for each B, B' the Hopf superalgebras $(\mathbb{C}[G_i] \times \wedge V_i)^{e^B}$ and $(\mathbb{C}[G_i] \times \wedge V_i)^{e^{B'}}$ are twist equivalent by twisting the comultiplication. Thus, by [AEG, Proposition 3.2.1], $A(G_i, V_i, u_i, B)$ and $A(G_i, V_i, u_i, B')$ are also twist equivalent by twisting the comultiplication. Thus their duals are quasi-isomorphic in our sense.

Etingof and Gelaki have shown in [EG, Corollary 4.3] that the families of duals contain infinitely many quasi-isomorphism classes of Hopf algebras.

The three families are the pointed Hopf algebras of liftings of biproducts corresponding to Γ, V as follows:

(F1) $\Gamma = C_8 = \langle g \rangle$, the cyclic group of order 8 with generator g and $\eta \in \widehat{\Gamma}$ with $\eta(g) = q$, q a primitive 8th root of unity. $V = Kx_1 \oplus Kx_2$ where $x_1 \in V_g^{\eta^4}$ and $x_2 \in V_{g^5}^{\eta^4}$.

(F2) $\Gamma = C_8 = \langle g \rangle$, and $\eta \in \widehat{\Gamma}$ as above. $V = Kx_1 \oplus Kx_2$ where $x_1 \in V_g^{\eta^4}$ and $x_2 \in V_{g^3}^{\eta^4}$.

(F3) $\Gamma = C_2 \times C_4$ where $C_2 = \langle g \rangle$ and $C_4 = \langle h \rangle$. Let $\eta \in \widehat{\Gamma}$ be defined by $\eta(g) = 1$, $\eta(h) = q$ where q is a primitive 4th root of unity. $V = Kx_1 \oplus Kx_2$ where $x_1 \in V_h^{\eta^2}$ and $x_2 \in V_{gh}^{\eta^2}$.

Example 5.11. Let $A = B(V) \# K[\Gamma]$ for any of the V, Γ in (F1)–(F3). Let H denote $K[\Gamma]$ and let $a_1, a_2, a \in K$. Define $\gamma = \gamma(a_1, a_2, a) : A \otimes A \rightarrow K$ by $\gamma(x_1^i x_2^j \otimes x_1^k x_2^l) = 0$ if $i + j \neq k + l$,

$$\gamma(1 \otimes -) = \gamma(- \otimes 1) = \varepsilon,$$

$$\gamma(x_i \otimes x_i) = a_i, \quad \gamma(x_2 \otimes x_1) = a, \quad \gamma(x_1 \otimes x_2) = 0,$$

$$\gamma(x_1 x_2 \otimes x_1 x_2) = -a_1 a_2,$$

and extend γ to $A \otimes A$ by H -bilinearity. Then by Proposition 5.9, $\gamma \in Z_H^2(A, K)$ and $A^\gamma \cong A(a_1, a_2, a)$.

We note that for γ to be a cocycle we must have that $\gamma(x_1 x_2 \otimes x_1 x_2) = -a_1 a_2$; to see this, apply (21) to the triple $x_1, x_2, x_1 x_2$.

Remark 5.12. In general, for $\gamma \in Z_H^2(A, K)$, the convolution inverse of γ will be a cocycle for A^γ , but need not be a cocycle for A . In Example 5.11, however, γ^{-1} is also a cocycle for A with the scalars a_i, a replaced by their negatives.

Remarks 5.13. (i) Let $B := A(a_1, a_2, a)$ be a lifting of the Radford biproduct $A = \mathcal{B}(V) \# K[\Gamma]$, from Example 5.11. Then B gives a splitting datum $(B \cong R \#_\xi H, H, \pi_B, \sigma)$ where σ is inclusion, $H = K[\Gamma]$ and $R = B^{co\pi_B}$ with the projection π_B as described in Example 3.4. From the definition of the cocycle ξ in 2.2.3, we have that (taking H -bilinearity into account) $\xi = \varepsilon \otimes \varepsilon$ except for the following:

$$\xi(x_i \otimes x_i) = a_i(1 - g_i^2), \quad \xi(x_2 \otimes x_1) = a(1 - g_1 g_2),$$

$$\xi(x_1 x_2 \otimes x_1 x_2) = \pi(x_1(-x_1 x_2 + a(1 - g_1 g_2))x_2) = -a_1 a_2(1 - g_1^2)(1 - g_2^2).$$

By (9), the inverse to ξ is given by

$$\xi^{-1} = -\xi \quad \text{on } R_i \otimes R_j \text{ for } i + j < 4 \quad \text{and} \quad \xi^{-1}(x_1 x_2 \otimes x_1 x_2) = \xi(x_1 x_2 \otimes x_1 x_2).$$

Since the image of the cocycle ξ is in the centre of B , then $R = B^{co\pi_B}$ is associative.

(ii) For the cocycle γ defined in Example 5.11, $R \#_\xi H \cong \mathcal{B}(V)^\gamma \#_{\varepsilon_\gamma} H$, so that we have splitting data $(R \#_\xi H, H, \pi_B, \sigma)$ and $(\mathcal{B}(V)^\gamma \#_{\varepsilon_\gamma} H, H, \pi_A, \sigma)$. Since $\gamma^{\pm 1}(x_1 \otimes x_2) = 0$, then $m_A(x_1 \otimes x_2) = m_{A^\gamma}(x_1 \otimes x_2)$ and $\pi_A = \pi_B$, $\mathcal{B}(V)^\gamma = R$, and $\xi = \varepsilon_\gamma = \gamma_R * (H \otimes \gamma_R^{-1})\rho_{R \otimes R}$.

Let Λ denote the total integral from H to K ; $\Lambda(g) = \delta_{g,1}$. Then

$$\Lambda \circ \xi = \Lambda \circ \varepsilon_\gamma = \Lambda \circ (\gamma_R * (H \otimes \gamma^{-1})\rho_{R \otimes R}) = (\gamma_R * (\Lambda \otimes \gamma^{-1})\rho_{R \otimes R}).$$

In family (F1), $\rho_{R \otimes R}(z) = 1 \otimes z$ only when $z = 1 \otimes 1$. Thus here $\Lambda \circ \pi \circ m_B = \Lambda \circ \xi = \gamma_R$.

However in families (F2) and (F3), we have $\rho_{R \otimes R}(z) = 1 \otimes z$ for $z = 1 \otimes 1$ and also for $z = x_1 x_2 \otimes x_1 x_2$ so that

$$\Lambda \circ \xi(x_1 x_2 \otimes x_1 x_2) = \gamma_R(x_1 x_2 \otimes x_1 x_2) + \gamma_R^{-1}(x_1 x_2 \otimes x_1 x_2) = 2\gamma_R(x_1 x_2 \otimes x_1 x_2).$$

One can also see this directly by applying Λ to ξ as described above.

(iii) In Example 5.10, $(\Lambda \otimes \gamma^{-1}) \circ \rho_{R \otimes R}$ is nonzero only on $1 \otimes 1$ since $\rho_{R \otimes R}(x_1^i x_2^j \otimes x_1^k x_2^m) = c^{i+j+k+m} \otimes x_1^i x_2^j \otimes x_1^k x_2^m$ and $i + j + k + m \leq 8$. Thus $\Lambda \circ \xi = \gamma$.

Even though $\Lambda \circ \xi$ might not be a cocycle for R above, $\Lambda \circ \xi$ is still a left H -linear map since Λ is ad-invariant and ξ is left H -linear with respect to the adjoint action. The next lemma will apply to this situation. Coalgebras and the braiding in the category ${}^H_H\mathcal{YD}$ are described in Section 2.1.

Lemma 5.14. For H a Hopf algebra, let R be a coalgebra in ${}^H_H\mathcal{YD}$. Let c be the braiding in ${}^H_H\mathcal{YD}$. Suppose that x, y are elements of R such that

- (i) $c\Delta_R(x) = \Delta_R(x)$ and $c\Delta_R(y) = \Delta_R(y)$,
(ii) $(R \otimes c^2 \otimes R)(\Delta_R \otimes \Delta_R)(x \otimes y) = (\Delta_R \otimes \Delta_R)(x \otimes y)$.

Let $\omega \in \text{Hom}(R \otimes R, K)$ be left H -linear and let $\mu : R \otimes R \rightarrow R$ be a linear map. Then

$$(\omega * \mu)(x \otimes y) = (\mu * \omega)(x \otimes y).$$

Proof. Set $z := x \otimes y$. Recall that $c_{R \otimes R, R \otimes R} = (R \otimes c \otimes R)(c \otimes c)(R \otimes c \otimes R)$ and thus

$$\begin{aligned} z_{(-1)}^{(1)} \cdot z_{(0)}^{(2)} \otimes z_{(0)}^{(1)} &= c_{R \otimes R, R \otimes R}(z_{(-1)}^{(1)} \otimes z_{(0)}^{(2)}) \\ &= (R \otimes c \otimes R)(c \otimes c)(R \otimes c \otimes R)(z_{(-1)}^{(1)} \otimes z_{(0)}^{(2)}) \\ &= (R \otimes c \otimes R)(c \otimes c)(R \otimes c \otimes R)(R \otimes c \otimes R)(\Delta_R(x) \otimes \Delta_R(y)) \\ &= (R \otimes c \otimes R)(c \otimes c)(\Delta_R(x) \otimes \Delta_R(y)) \\ &= (R \otimes c \otimes R)(c\Delta_R(x) \otimes c\Delta_R(y)) \\ &= (R \otimes c \otimes R)(\Delta_R(x) \otimes \Delta_R(y)) = z_{(-1)}^{(1)} \otimes z_{(0)}^{(2)}. \end{aligned}$$

Hence

$$\begin{aligned} (w * \mu)(z) &= w(z_{(-1)}^{(1)})\mu(z_{(0)}^{(2)}) = w(z_{(-1)}^{(1)} \cdot z_{(0)}^{(2)})\mu(z_{(0)}^{(1)}) \\ &= \varepsilon_H(z_{(-1)}^{(1)})w(z_{(0)}^{(2)})\mu(z_{(0)}^{(1)}) = w(z_{(0)}^{(2)})\mu(z_{(0)}^{(1)}) \\ &= (\mu * w)(z). \quad \square \end{aligned}$$

For example, if $V = Kx \oplus Ky$ is a quantum linear plane and $R = \mathcal{B}(V)$, then the conditions of Lemma 5.14 apply to x, y with $\mu = m_R$. In the examples of dimension 32 in this section, c^2 is the identity on $R \otimes R$ and $c\Delta_R = \Delta_R$. If ω is convolution invertible and left H -linear, then

$$\omega * m_R * \omega^{-1} = m_R.$$

In particular, ω could be $\Lambda \circ \xi$.

5.3. Some general remarks

Given a general splitting datum $(A = R \#_{\xi} H, H, \pi, \sigma)$, one problem is to find $\omega \in Z_H^2(A, K)$ such that $(A^{\omega}, H, \pi, \sigma)$ is a trivial splitting datum, in other words, such that ξ_{ω_R} is trivial.

As in Remarks 5.13, from the definition in Theorem 4.11, if $\xi_{\omega_R} = \varepsilon$, then:

$$\xi = u_H \omega_R^{-1} * (H \otimes \omega_R) \rho_{R \otimes R} = u_H \omega_R^{-1} * \Psi(\omega_R),$$

and then for any $f \in \text{Hom}(H, K)$,

$$f \circ \xi = \omega_R^{-1} * (f \otimes \omega_R)(\rho_{R \otimes R}),$$

so that

$$f \circ \xi = \omega_R^{-1} \quad \text{if and only if} \quad (f \otimes \omega_R) \rho_{R \otimes R} = \varepsilon_R \otimes \varepsilon_R. \quad (35)$$

Similarly

$$f \circ \xi^{-1} = \omega_R \quad \text{if and only if} \quad (f \otimes \omega_R^{-1})\rho_{R \otimes R} = \varepsilon_R \otimes \varepsilon_R. \quad (36)$$

Even though we know that $\Psi(\omega_R^{-1}) = (H \otimes \omega_R^{-1})\rho_{R \otimes R}$ is the convolution inverse to $\Psi(\omega_R) = (H \otimes \omega_R)\rho_{R \otimes R}$, it is not clear if the equalities in (35) and (36) are equivalent.

If f above is an integral λ for H , we know from the examples of dimension 32 that $\lambda \circ \xi$ is not always a cocycle. Nevertheless, if it is a cocycle, then twisting by $\lambda \circ \xi$ yields a trivial splitting datum.

Proposition 5.15. *Let (A, H, π, σ) be a splitting datum with associated pre-bialgebra (R, ξ) . Let λ be a left integral for H in H^* . Then*

$$\xi * [(H \otimes \lambda\xi)\rho_{R \otimes R}] = u_H \lambda \circ \xi.$$

For $A = R \#_{\xi} H$, if $\gamma \in Z_H^2(A, K)$ such that $\lambda \circ \xi = \gamma_R^{-1}$, then ξ_{γ_R} is trivial and $(R \#_{\xi} H)^{\gamma} = R^{\gamma_R} \# H$.

Proof. We have

$$\xi * [(H \otimes \lambda\xi)\rho_{R \otimes R}] = (m_H \otimes \lambda\xi)(\xi \otimes \rho_{R \otimes R})\Delta_{R \otimes R} \stackrel{(2)}{=} (H \otimes \lambda)\Delta_H \xi = u_H \lambda \xi.$$

If $\gamma \in Z_H^2(A, K)$ such that $\lambda\xi = \gamma_R^{-1}$, then $\xi * (H \otimes \gamma_R^{-1})\rho_{R \otimes R} = \xi * (H \otimes \lambda\xi)\rho_{R \otimes R} = u_H \lambda \xi = u_H \gamma_R^{-1}$ so that

$$\xi_{\gamma_R} = u_H \gamma_R * \xi * (H \otimes \gamma_R^{-1})\rho_{R \otimes R} = u_H \gamma_R * u_H \gamma_R^{-1} = u_H \varepsilon_{R \otimes R}. \quad \square$$

The properties of $\lambda\xi$ will be investigated in a forthcoming paper.

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References

- [AEG] N. Andruskiewitsch, P. Etingof, S. Gelaki, Triangular Hopf algebras with the Chevalley property, *Michigan Math. J.* 49 (2) (2001) 277–298.
- [AS1] N. Andruskiewitsch, H.-J. Schneider, On the classification of finite-dimensional pointed Hopf algebras, *Ann. of Math.* 171 (1) (2010) 375–417.
- [AS2] N. Andruskiewitsch, H.-J. Schneider, Lifting of quantum linear spaces and pointed Hopf algebras of order p^3 , *J. Algebra* 209 (1998) 658–691.
- [AM] A. Ardizzoni, C. Menini, Small bialgebras with a projection: Applications, *Comm. Algebra* 37 (8) (2009) 2742–2784.
- [AMSte] A. Ardizzoni, C. Menini, D. Stefan, A monoidal approach to splitting morphisms of bialgebras, *Trans. Amer. Math. Soc.* 359 (2007) 991–1044.
- [AMStu] A. Ardizzoni, C. Menini, F. Stumbo, Small bialgebras with projection, *J. Algebra* 314 (2) (2007) 613–663.
- [B] M. Beattie, An isomorphism theorem for Ore extension Hopf algebras, *Comm. Algebra* 28 (2) (2000) 569–584.
- [BDG] M. Beattie, S. Dăscălescu, L. Grünenfelder, Constructing pointed Hopf algebras by Ore extensions, *J. Algebra* 225 (2) (2000) 743–770.
- [Doi] Y. Doi, Braided Hopf algebras and quadratic bialgebras, *Comm. Algebra* 21 (5) (1993) 1731–1749.
- [EG] P. Etingof, S. Gelaki, On families of triangular Hopf algebras, *Int. Math. Res. Not. IMRN* (14) (2002) 757–768.
- [G] M. Graña, Pointed Hopf algebras of dimension 32, *Comm. Algebra* 28 (6) (2000) 2935–2976.
- [GM] L. Grünenfelder, M. Mastnak, Pointed and copointed Hopf algebras as cocycle deformations, preprint, arXiv:0709.0120v2.
- [K] C. Kassel, *Quantum Groups*, Grad. Texts in Math., vol. 155, Springer, New York, 1995.
- [Mas1] A. Masuoka, Abelian and non-abelian second cohomologies of quantized enveloping algebras, *J. Algebra* 320 (1) (2008) 1–47.
- [Mas2] A. Masuoka, Defending the negated Kaplansky conjecture, *Proc. Amer. Math. Soc.* 129 (2001) 3185–3192.

- [Mo] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, CBMS Reg. Conf. Ser. Math., vol. 82, Amer. Math. Soc., Providence, RI, 1993.
- [Rad] D.E. Radford, The structure of Hopf algebras with a projection, *J. Algebra* 92 (1985) 322–347.
- [Scha] P. Schauenburg, The structure of Hopf algebras with a weak projection, *Algebr. Represent. Theory* 3 (2000) 187–211.
- [SvO] D. Ştefan, F. van Oystaeyen, The Wedderburn–Malcev theorem for comodule algebras, *Comm. Algebra* 27 (1999) 3569–3581.
- [Sw] M. Sweedler, *Hopf Algebras*, Benjamin, New York, 1969.